A GENERAL MODEL FOR FREE-RESPONSE DATA

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CHAPTER 1

INTRODUCTION

One of the important topics in psychological measurement is how to make a specified psychological behavior or process measurable, and in this context the latent trait model [Lord & Novick, 1968] may be a most useful mathematical model. For a relatively simple psychological process its physiological counterpart may easily be clarified, but in most cases this is not true, at the present stage of development. Thus the psychological scaling is usually the device for measuring a hypothetical construct or latent trait, rather than for clarifying the relationship between psychological and physiological behaviors.

The latent trait model has been developed both as a mental test theory and as a social psychological measurement, in which we mainly consider one psychological dimension, and deal with a Bernoulli trial for an item response. The use of item characteristic function was initiated by Ferguson [Ferguson, 1942], and has been elaborated by Lawley [Lawley, 1943], Tucker [Tucker, 1946], Lord and others [Lord, 1952; Lord & Novick, 1968] in the mental test theory. Birnbaum [Birnbaum, 1968] effectively utilized the logistic model of the dichotomous item as a substitute for the normal ogive model to answer many problems in test construction and the scoring of responses. In social psychology, Lazarsfeld [Lazarsfeld, 1959], developed the latent structure analysis, using the trace line which is fundamentally equivalent to the item characteristic function in the mental test theory. Recently Samejima [Samejima, 1969] suggested a more generalized model, in which graded item responses are treated as well as dichotomous item responses, and the response pattern of graded scores is used in estimation of the latent trait.

The main objective of the present paper is: 1) to propose a general model for free-response data collected for measuring a specified unidimensional psychological process, by expanding the latent trait model further to include the case in which an item is not scored; 2) to systematize situations which vary with respect to the scoring level of items; 3) to find out general conditions for the operating characteristic of an item response category to provide a unique maximum likelihood estimator. Under this model the psychological construct can be a perceptual sensitivity, potentiality for learning mazes, a personality trait, a mental ability, an attitude, etc., which distributes almost in all fields of psychology.

1.1 Free-Response Data

Since this is a general model for free-response data, it may be worth questioning what free-response data in psychological studies are. By free-response data we usually mean a set of data collected by using the free-response format of questioning, to which the subject should find his answer out of all conceivable responses. As distinct from this, a multiple-choice item presents a certain limited number of prearranged choices or alternatives, from which the subject should choose one.

Thus in usual cases the definition is made in terms of the format of questioning, rather than in terms of the content or quality of data. It is easily seen, however, that in many cases the whole space of possible answers to an item consists of a relatively small number of elements, even if they are not explicitly suggested as choices or alternatives. To give a simple and extreme example, suppose the subject is asked to answer whether he has perceived a light spot on the screen in a dark room, after each presentation or non-presentation of the light. In this example, the whole space consists of only two elements, "have seen" and "have not seen", even if the format of questioning used is that of the free-response situation. In other words, it will give little effect on the result if we change the format of questioning into that of the multiple-choice situation, so that the subject should be asked to check one of the two alternatives given. An important implication is, however, that the probability with which the subject answers correctly by guessing can be large, when the whole space consists of a small number of possible answers, even if a question is asked with the free-response format of questioning.

In contrast to this, suppose that the whole space is an infinite set of possible answers. Thus it is impossible to make a multiple-choice item out of it, unless we take a finite subset of possible answers as our alternatives. In such a case, the quality of the item is inevitably changed by the use of the multiple-choice format.

The above brief discussion suggests that for scientific analysis of data we should strictly distinguish the free-response content from the free-response format, and also the multiple-choice content from the multiple-choice format. *We must define our free-response data with respect to their contents,* rather than with respect to their formats of questioning with which they were collected. The possibility of obtaining free-response data with the multiple-choice format is discussed elsewhere [Samejima, 1968; in preparation [b]].

Thus in this paper we deal with any data with valid contents, regardless of the format of questioning adopted. In so doing some degree of tolerance may be considered so that categorized data are also included, if the way of categorization is appropriate enough not to invite too much noise. This categorization can be made afterwards, as well as beforehand.

1.2 Basic Concepts and Assumptions

In this paper, we shall call a specified psychological construct the *trait,* or the *latent trait,* whether it is customarily called the trait or not. Let θ

denote the trait. The trait can operationally be defined in terms of a set of n *items*, to which specific psychological behaviors are elicited.

The first general assumption throughout this paper is the following.

(1) The latent trait is *uni-dimensional, i.e.*, the n items have only one psychological dimension in common. To be more precise, using the additive model, each item variable is considered as a linear combination of one common factor and a unique factor.

Let g, h, or j denote an item, and k_g, k_h, or k_j be a specific item response, or a behavior elicited by the item. In actual situations these responses to a specified item may be more or less categorized. Especially in the free-response situation each item may elicit a great many different responses, and it is neither possible nor meaningful to treat them separately. Thus we usually analyze data by categorizing them more or less in terms of their identities. In the example given in Section 1.1, for instance, the subject's free answers are classified into two categories, "have seen a light spot", and "have not seen a light spot". We shall call such a category the *item response category*, or the *response category*, and use it in preference to "item response", and for simplicity let k_g, k_h, or k_j denote the item response category also. Thus these symbols represent an appropriately defined event class of the whole space of all possible answers to a specified item each. The set of all the response categories to a specified item can either be finite or enumerable, and the categories should be disjoint and exhaustive in the whole space.

We shall call this general situation the *nominal response level* or simply the *nominal level*. If all the response categories to an item can be arranged in a rank order so that they provide score categories, we shall call this special case the *graded response level*, or simply the *graded level*. In this case, if the lowest category is specified, we shall assign non-negative integers, 0 through m_g, reversely to the categories arranged in the rank order, which will be called *item scores*. Any categorical judgment data in a well designed psychological experiment or survey can be regarded as a good example of the graded level situation, in which m_g is a finite number. If $m_g = 1$ in the graded level situation, it will be called the *dichotomous response level*, or simply the *dichotomous level*. The situation is common in mental measurement, in which each item is scored either correct or incorrect. Also the example in Section 1.1 is another typical example of the dichotomous level situation.

When we have n items for measuring the trait, we obtain a sequence of n responses as the results of the subject's performance. By the *response pattern* we mean a sequence of specified item response categories given by the subject or respondent to a set of n items. Let V denote the response pattern, which is given by a vector such that

$$V = (k_1, k_2, \cdots, k_g, \cdots, k_n) \tag{1-2-1}$$

on the nominal level, and

(1-2-2)
$$V = (x_1, x_2, \cdots, x_g, \cdots, x_n)$$

on the graded level. Then our second general assumption is the following.

(2) The *principle of local independence* [Lord & Novick, 1968, pages 360–362] or conditional independence, holds, *i.e.*, for a fixed value of the trait the distributions of the item response categories are independent of one another.

We shall call the probability of a specified event, which is defined for a fixed value of the trait, the *operating characteristic*. Let $P_{k_g}(\theta)$, $P_{x_g}(\theta)$, and $P_V(\theta)$ denote the operating characteristics of item response categories k_g and x_g, and of response pattern V respectively. Since the operating characteristic is defined for a fixed value of the trait, it is a function of θ. Thus the general assumption (2) can be expressed by the formula

(1-2-3)
$$P_V(\theta) = \prod_{k_g \in V} P_{k_g}(\theta)$$

on the nominal level, and

(1-2-4)
$$P_V(\theta) = \prod_{x_g \in V} P_{x_g}(\theta)$$

on the graded level.

The third general assumption is:

(3) The operating characteristic of an item response category is *three-times differentiable* with respect to θ.

The *basic function* $A_{k_g}(\theta)$ [Samejima, 1969, page 24] is defined by

(1-2-5)
$$A_{k_g}(\theta) = \frac{\partial}{\partial\theta} \log P_{k_g}(\theta) = \frac{\partial}{\partial\theta} P_{k_g}(\theta) / P_{k_g}(\theta)$$

for a specified item response category k_g, with the two limits

(1-2-6)
$$\begin{cases} C_{k_g,\underline{\theta}} = \lim_{\theta \to \underline{\theta}} A_{k_g}(\theta) \\ C_{k_g,\bar{\theta}} = \lim_{\theta \to \bar{\theta}} A_{k_g}(\theta), \end{cases}$$

where $\underline{\theta}$ and $\bar{\theta}$ are the lower and upper bounds of the range of θ. On the graded response level, the subscript k_g is replaced by x_g in the above three formulas.

Finally, our fourth general assumption is the following.

(4) The upper and lower bounds of the trait are positive and negative

infinities, *i.e.*, we have

(1-2-7) $$-\infty < \theta < \infty .$$

Since in some cases we can reasonably assume that one, or both, of the upper and lower bounds of the trait is finite, this fourth assumption may not appear reasonable. Such a situation can be regarded, however, as the case in which either the range of θ is truncated, or the variable is transformed into another, one being strictly increasing in the other. To give an example for the latter case, let θ be transformed into τ by

(1-2-8) $$\tau = c_1/(1 + c_2 e^{-\theta})$$

where c_1 and c_2 are positive constants. In this example, we have

(1-2-9) $$0 < \tau < c_1$$

for the range of τ. When $c_1 = 1$, the trait takes only positive values less than unity. In any case, we have for the maximum likelihood estimator $\hat{\tau}_V$

(1-2-10) $$\hat{\tau}_V = \tau(\hat{\theta}_V)$$

if the maximum likelihood estimator $\hat{\theta}_V$ exists, by virtue of the transformation-free character of the maximum likelihood estimator [Samejima, 1969, pages 6–7].

CHAPTER 2

SYNDROME RESPONSE PATTERNS

As was observed in Chapter 1, the operating characteristic is the sole function which relates the item response category, k_g, with the trait, θ, in the present model. In order to estimate the subject's position on the trait dimension from his response pattern on n items, its operating characteristic takes an important role either in the maximum likelihood estimation or in Bayesian estimations [Samejima, 1969, Chapters 2 & 7]. When the distribution of the trait is unknown, maximum likelihood estimation will be the most reasonable method. In order for the operating characteristic of a response pattern to provide a unique maximum, however, the operating characteristic should be a uni-modal function of θ, or a strictly monotone function of θ if we permit a terminal maximum. Thus, in short, it should have a relatively simple relationship with the trait.

For reasons discussed by Samejima [Samejima, 1969, pages 9 & 10], conditions which provide such an operating characteristic of the response pattern should be considered with respect to a single item, rather than with respect to a specified set of n items. A sufficient condition proposed by Samejima is the joint satisfaction of Conditions (i) and (ii)* such that

(2-1) $$\text{Condition (i):} \quad \frac{\partial}{\partial\theta} A_{k_g}(\theta) \leq 0$$

where an equality should hold at most at an enumerable points of θ, *i.e.*, the basic function should be strictly decreasing in θ, and

(2-2) $$\text{Condition (ii)*:} \quad \begin{cases} C_{k_g, \underline{\theta}} \geq 0 \\ C_{k_g, \bar{\theta}} \leq 0 \end{cases},$$

where one at least is a strict inequality.

By virtue of the general assumption (4) given in Section 1.2, Condition (i) automatically involves Condition (ii)* under the present assumptions. To prove this, it is easily seen from (1-2-5) and (2-1) that $P_{k_g}(\theta)$ should be either a uni-modal function of θ or a strictly monotone function of θ. If it is uni-modal, strict inequalities should hold in both formulas of (2-2) in order for Condition (i) to be true. If it is a monotone function, one of the formulas of (2-2) should equal zero, since $P_{k_g}(\theta)$ is a bounded function of θ, and it follows that a strict inequality should hold in the other, in order for Condition (i) to be true. Thus Condition (i) automatically includes Condition (ii)* under the present general assumptions.

Suppose that a response pattern consists of item response categories, for each of which Condition (i) is true. We shall call such a response pattern a *syndrome response pattern,* in the sense that it necessarily provides a unique local maximum or a terminal maximum, and, therefore, can be a good indicator of the subject's position on the trait dimension. When $n = 1$, *i.e.*, there is only one item, a response pattern consists of a single item response category. If it is a syndrome response pattern, this single response category satisfies Condition (i). To generalize the term, we shall call any item response category which satisfies Condition (i) under the present general assumptions a *syndrome response category.*

It is obvious from the definition of the syndrome response pattern that it has additive properties [Cramér, 1946, page 10]. Let the whole space S be the set of a finite or enumerable syndrome response categories, each of which is provided by a different item. Let D denote a class, the syndrome response pattern. From this we have:

1) The whole space S belongs to the class D.
2) If every set of the sequence S_1, S_2, $\cdots$ belongs to D, both unions and intersections of these sets also belong to D.
3) If S_1 and S_2 belong to D, and S_2 is a subset of S_1, then the difference, $S_1 - S_2$, belongs to D.

Thus the syndrome response pattern, D, is an additive class.

From the definition of the syndrome response category, we can see that its operating characteristic takes either one of the following three types.

Type (i): A monotone increasing function of θ with zero and a positive value less than or equal to unity as its lower and upper asymptotes respectively, and whose first derivative should be positive for all θ, *i.e.*, it should not equal zero at any point of θ.

Type (ii): A monotone decreasing function of θ with a positive value less than or equal to unity as its upper asymptote and zero as its lower asymptote, and whose first derivative should be negative for all θ, *i.e.*, it should not equal zero at any point of θ.

Type (iii): A uni-modal function of θ with zero as its two lower asymptotes, and whose first derivative should be positive up to the modal point and should be negative afterwards.

These three types of operating characteristic are simple and meaningful, in the sense that they can be direct indicators of the subjects' positions on the trait continuum. If, for instance, $P_{k_g}(\theta)$ is constant throughout the whole range of θ, the relationship between the response category and the trait is simple enough, but not meaningful, since it does not provide any information about the subject's position on the trait dimension. If $P_{k_g}(\theta)$ has more than one modal point, we must say their relationship is by no means simple.

We must note, however, that the satisfaction of one of these three statements is a necessary condition that k_g should be a syndrome response category, but not a sufficient condition. We can easily conceive, for example, of a bi-modal operating characteristic of response pattern as the product of two operating characteristics of item response categories of Type (iii), which are not syndrome response categories and whose modal points are substantially far from each other. Thus the resultant operating characteristic of the response pattern cannot be a syndrome to the subject's position on the trait continuum, though each operating characteristic of the item response category is a *symptom* of the subject's position by itself. In a reversed way, two or more complicated operating characteristics of item response category can provide a *symptom* operating characteristic of the response pattern. Figure 2-1 illustrates with an example of such a case, in which two bi-modal operating characteristics of item response category (dotted and dashed lines) provide a uni-modal operating characteristic of response pattern (solid line) according to the principle of local independence.

The syndrome response pattern and category must be considered distinctly from the cases illustrated above, and will be our main concerns

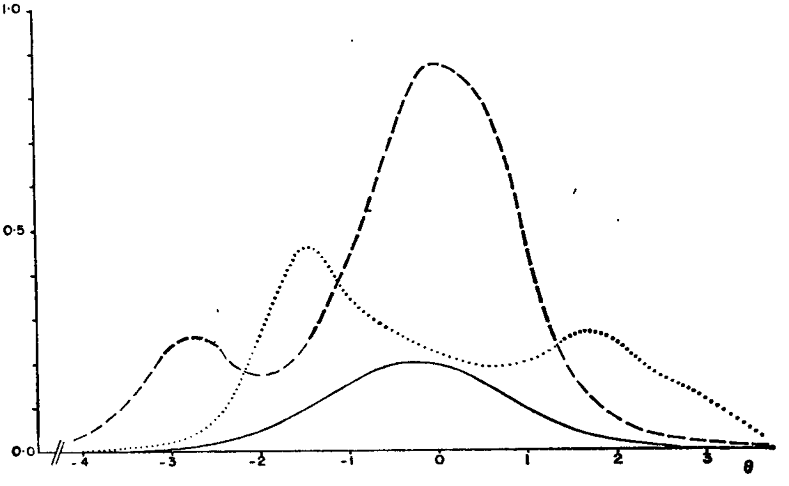

FIGURE 2–1

An example, in which two bi-modal operating characteristics of item response category (dotted and dashed lines) provide a uni-modal operating characteristic of response pattern (solid line) according to the principle of local independence.

in the subsequent chapters, when conditions which provide a unique maximum are discussed on different levels. We must keep in mind that, by virtue of the transformation-free character of the maximum likelihood estimator, a unique maximum condition assures us its existence with respect to θ, as well as with respect to any variable which is a continuous and one-to-one mapping of θ.

CHAPTER 3

THE NOMINAL RESPONSE LEVEL

As was introduced in Section 1.2, the nominal response level is the most general situation, of which both the graded response level and the dichotomous response level are subsets. Distinct from the usual latent trait theory where discussion is solely directed towards dichotomous items, the item response category is the focus of our attention throughout the rest of this paper, rather than the item itself as the smallest unit.

3.1 Assumptions and Formulas

All the discussion on the nominal response level starts with the assumption that the factors affecting the respondent's attitude towards a specified response category k_g can be classified into *two distinct tendencies*, "being attracted by k_g" and "its rejection". In other words, our assumption is: if the subject chose a specified response category k_g as his answer, it means that he was attracted by that category, and its rejection did not occur to him; if, on the other hand, the subject did not choose k_g , it means either he was not attracted by that category, or he was, but eventually rejected it.

Let $R_{k_g}(\theta)$ denote the probability, with which the subject of trait θ is attracted by the response category k_g , and $U_{k_g}(\theta)$ be the conditional probability with which he rejects the category, given that he has already been attracted by k_g . These two probabilities are defined for a fixed value of θ, and, therefore, are functions of θ. On the assumption made in the preceding paragraph, the fundamental formula for the operating characteristic of the item response category on the nominal level is given by

$$P_{k_g}(\theta) = R_{k_g}(\theta)[1 - U_{k_g}(\theta)]. \tag{3-1-1}$$

Thus $P_{k_g}(\theta)$ can be a function of various types depending upon the functional formulas of $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$, with the constraint

$$\sum_{k_g} P_{k_g}(\theta) = 1. \tag{3-1-2}$$

In an extreme case where $R_{k_g}(\theta) = 0$ for all the values of θ, *i.e.*, the probability with which the subject is attracted by k_g is zero regardless of his position on the trait dimension, the operating characteristic of k_g is zero for the entire range of θ. We can conceive of all sorts of dummy answers to an item as examples of such a situation. In another extreme case where $U_{k_g}(\theta) = 1$ for all the values of θ, $P_{k_g}(\theta)$ is also zero throughout the whole range of θ, since the category is rejected with probability one regardless of the value of $R_{k_g}(\theta)$. If $U_{k_g}(\theta) = 0$ for all θ, the conditional probability of rejection of k_g is zero, and the operating characteristic is totally determined by $R_{k_g}(\theta)$. The correct answer to a power test item may be a good example of this situation. Similarly, if $R_{k_g}(\theta) = 1$ for all θ, the operating characteristic is totally determined by the second factor of (3-1-1), *i.e.*, $[1 - U_{k_g}(\theta)]$. If $R_{k_g}(\theta) = 1$ and $U_{k_g}(\theta) = 0$ for all θ, $P_{k_g}(\theta) = 1$ throughout the entire range of θ, which is of little use in reality.

The second assumption on the nominal response level is that both $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ are *three-times differentiable* with respect to θ throughout the whole range of θ. It is easily seen from (3-1-1) that this assumption is consistent with the general assumption (3) given in Section 1.2.

The basic function of the item response category k_g on the nominal level is obtained from (1-2-5) and (3-1-1) such that

$$\text{(3-1-3)} \qquad \begin{aligned} A_{k_g}(\theta) &= \frac{\partial}{\partial\theta} \log R_{k_g}(\theta) + \frac{\partial}{\partial\theta} \log [1 - U_{k_g}(\theta)] \\ &= \frac{\partial}{\partial\theta} R_{k_g}(\theta)/R_{k_g}(\theta) - \frac{\partial}{\partial\theta} U_{k_g}(\theta)/[1 - U_{k_g}(\theta)]. \end{aligned}$$

One characteristic of the nominal response level is that the functional formulas for $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ for a specified response category k_g *can be independent* from each other, and also they can be independent from those of other response categories to the same item g, with the sole constraint (3-1-2). Thus two or more non-scored response categories can be compared with one another on the trait continuum solely in terms of their operating characteristics a posteriori.

3.2 Sufficient Condition for k_g to be a Syndrome Response Category

A sufficient, though not necessary, condition that k_g should be a syndrome response category is the joint satisfaction of:

$$\text{(3-2-1)} \qquad \begin{cases} \dfrac{\partial^2}{\partial\theta^2} \log R_{k_g}(\theta) \leq 0 \\[2ex] \dfrac{\partial^2}{\partial\theta^2} \log [1 - U_{k_g}(\theta)] \leq 0 \end{cases}$$

for the whole range of θ, where an equality holds at most at an enumerable number of points of θ, as is easily seen from (2-1) and (3-1-3).

The fact that (3-2-1) is not a necessary condition can be observed through an example. If $R_{k_g}(\theta)$ is a strictly increasing function of θ with some positive value as its lower asymptote, it does not satisfy the first inequality of (3-2-1) for a certain range of θ. And yet it can provide a syndrome response category if it is combined with an adequate $U_{k_g}(\theta)$.

It is worth noting, however, that, as far as $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ satisfy (3-2-1), their functional formulas can be defined independently from each other, in order to make k_g a syndrome response category. We can conceive, moreover, of families of formulas expanded from $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ such that

$$\begin{cases} R^*_{k_g}(\theta) = R_{k_g}(\alpha_R\theta + \beta_R) \\ U^*_{k_g}(\theta) = U_{k_g}(\alpha_U\theta + \beta_U) \end{cases}, \tag{3-2-2}$$

where α_R and α_U are positive constants and β_R and β_U are real constants, any pair of members of which provides a syndrome response category for k_g , provided that (3-2-1) is true.

(3-2-1) can be expanded to permit one of the formulas to be equal to zero for some intervals of θ, with the constraint that $R_{k_g}(\theta) \neq 0$ or $U_{k_g}(\theta) \neq 1$. When either one of $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ is constant for all θ, the satisfaction of the compensating formula of (3-2-1) is the necessary and sufficient condition that k_g should be a syndrome response category. It is easily seen that an equality holds in one of the formula of (3-2-1) for the entire range of θ if, and only if, $R_{k_g}(\theta)$ or $U_{k_g}(\theta)$ is constant, since both are bounded functions.

A useful fact is that a set of $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ which satisfies (3-2-1) in its original form can be provided by $W(\theta)$, any three-times differentiable and strictly increasing function of θ with zero and unity as its lower and upper asymptotes, which satisfies

$$\frac{\partial^2}{\partial\theta^2} \log \left[\frac{\partial}{\partial\theta} W(\theta)\right] \leq 0, \tag{3-2-3}$$

where a strict inequality holds for all θ except, at most, for an enumerable number of points of θ. (3-2-3) implies that the first derivative of $W(\theta)$ should not equal zero at any point of θ. In this case, $R_{k_g}(\theta)$ or $[1 - U_{k_g}(\theta)]$ can be given by either one of:

$$CW(\theta),$$

$$C[1 - W(\theta)],$$

and

$$C^*W_1(\theta)[1 - W_2(\theta)],$$

where W_1 and W_2 are possibly different functional formulas satisfying (3-2-3) each, C is an arbitrary positive constant less than or equal to unity, and C^* is an arbitrary positive constant with the constraint that it should not make the third term greater than unity for any θ. For we obtain from (3-2-3) that

$$\begin{cases} \frac{\partial^2}{\partial\theta^2} \log W(\theta) \leq 0 \\ \frac{\partial^2}{\partial\theta^2} \log [1 - W(\theta)] \leq 0 \end{cases}, \tag{3-2-4}$$

where a strict inequality holds for all θ except, at most, for an enumerable number of points of θ each, and the left hand sides of these two formulas are also the second derivatives of log $R_{k_g}(\theta)$ or log $[1 - U_{k_g}(\theta)]$ when the first and second terms given above are used, and so is the sum total of the left hand sides of them when the third term is used, if we use W_1 in the first formula and W_2 in the second of (3-2-4). The proof that (3-2-4) can be obtained from (3-2-3) is given in Appendix in a more general form, in which x is used instead of θ, and f is used in place of W, and the restriction that $W(\theta)$ should have zero and unity as its asymptotes is excluded.

In general, if we group two or more syndrome response categories together, the resulting category is not always a syndrome response category. Since it is likely to happen in practical situations that we should group many categories of minor significance into one category named "others", this can be a fatal defect in estimation of the trait on the nominal level. In the homogeneous case of the graded response level, however, we shall obtain a syndrome response category if we combine two or more adjacent score categories together, which will be observed in Chapter 5.

If all the response categories to item g are syndrome response categories, we can distinguish the following three situations, which hereafter will be called Situations A, B and C.

Situation A: There are *one* response category whose operating characteristic is of Type (i), *i.e.*, strictly increasing in θ, and *another* whose operating characteristic is of Type (ii), *i.e.*, strictly decreasing in θ. In addition to them, there possibly are one or more categories the operating characteristics of which are of Type (iii), *i.e.*, uni-modal.

Situation B: There are *more than one* response category whose operating characteristic is of Type (i) (or of Type (ii)), and *at least one* response category whose operating characteristic is of Type (ii) (or of Type (i)). In addition to them, there possibly are one or more categories the operating characteristics of which are of Type (iii).

Situation C: There are an *enumerable* number of categories whose operating characteristics are of Type (iii), but *no* categories the operating characteristics of which are of Type (i), *or* no categories the opearting characteristics of which are of Type (ii), *or* neither of them.

We can easily see that Situation C can be true only if the set of all the response categories to item g is enumerable whereas the other two can be true whether it is finite or enumerable. Discussions will mainly be focused on Situations A and B in following sections, since in practical situations the set is usually finite.

3.3 Plausibility Curve as a Typical Example of the Nominal Level Situation

As an example of a characteristically nominal situation, we shall consider a non-scored category in ability measurement. Suppose a certain response category to item g of a power test is an *incorrect, but plausible,* answer. We may reasonably assume that, if the examinee's ability is very low, it is likely that he does not even relate that category with the item at all, whereas he does if his ability is sufficiently high. Thus in such a case $R_{k_g}(\theta)$ can be considered as being a strictly increasing function of θ, if the response category k_g has some direct significance to the ability measured. The conditional probability with which the response category is rejected because of its incorrectness is also a function of ability, and again this probability, $U_{k_g}(\theta)$, can be considered as being a strictly increasing function of θ for such a response category.

There can be varieties of such false answers if examinees have responded to the item freely, without being forced to choose one of prearranged alternatives. It is conceivable that some of the incorrect answers may require high levels of ability while some others may not, some may be related strongly to the ability measured whereas some others may not, and so forth.

An objective measure of the plausibility of a specified false answer is its operating characteristic. We shall call any operating characteristic of a false answer, whose $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ satisfy these conditions given in a previous paragraph, the *plausibility curve* in metal measurement, *provided that* it is a syndrome response category. As the conditions suggest, a plausibility curve is necessarily of Type (iii), *i.e.*, uni-modal, defined in Chapter 2. A schematized hypothesis for the plausibility curve will be the following. An examinee may be attracted by a specified wrong, but plausible, answer to item g, but may fail in detecting its irrationality; the probability with which this happens is a function of ability, and it increases as ability increases, reaches maximum at a certain level of ability, and then decreases afterwards. The modal point of a plausibility curve can be a measure of the ability level which is required for an examinee to stay with the plausibility of the false answer. If an item provides such response categories, their plausibility curves will be powerful sources of information in estimating examinees' abilities. In other words, we can make use of specific wrong answers to an item as sources of information, as well as the correct answer. It is conceivable that two or more distinct false answers may have exactly the same modal points, or, moreover, exactly the same plausibility curves. In such a case, it is hard to order these categories, and yet each non-scored category can be an information source by itself.

For the purpose of illustration, Figure 3-3-1 presents an example of the plausibility curve, whose $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ satisfy (3-2-1) with strict inequalities, and are given by

$$(3\text{-}3\text{-}1)\qquad \begin{cases} R_{k_g}(\theta) = \dfrac{0.9}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\theta} e^{-(t^2/2)}\, dt \\ U_{k_g}(\theta) = 1 - [1 - e^{-1.7}][1 + e^{1.7(\theta-0.5)}]^{-1} \end{cases} .$$

It is indicated by the formulas that the upper asymptote of $R_{k_g}(\theta)$ is 0.9, and the lower asymptote of $U_{k_g}(\theta)$ is exp (-1.7), *i.e.*, approximately 0.183. The two dashed curves in Figure 3-3-1 are $R_{k_g}(\theta)$ and $[1 - U_{k_g}(\theta)]$ respectively, the dotted curve is $U_{k_g}(\theta)$, and the solid curve is the plausibility curve, $P_{k_g}(\theta)$. In this example $W(\theta)$ for generating $R_{k_g}(\theta)$ is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-(t^2/2)}\, dt$$

and the one for generating $U_{k_g}(\theta)$ is

$$[1 + e^{-1.7(\theta-0.5)}]^{-1}.$$

It is easily seen that both satisfy (3-2-3).

3.4 Multi-Correct and Multi-Incorrect Responses

Non-ordered multi-correct or multi-incorrect responses have been discussed with regard to bio-assay rather than psychology. It is a situation in which there exist more than one response category which has a strictly

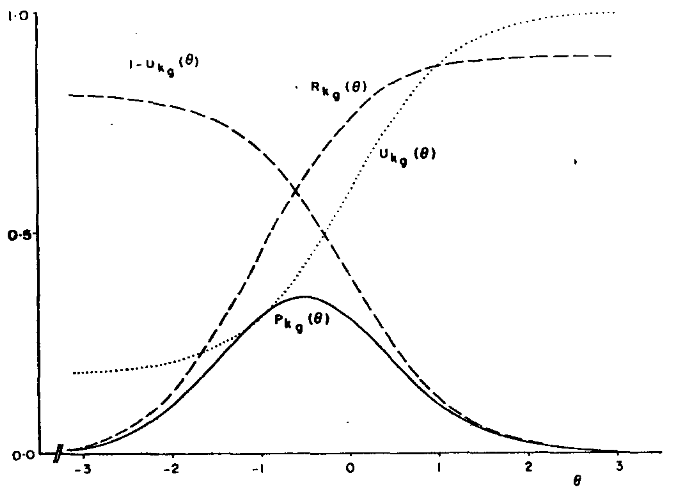

FIGURE 3–3–1

An example of the plausibility curve, whose $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ are given by (3-3-1). These formulas satisfy (3-2-1) together with strict inequalities so that the resulting response category is a syndrome response category.

increasing operating characteristic, or more than one whose operating characteristic is strictly decreasing in θ. Thus it corresponds to Situation B, if all the response categories to item g are syndrome response categories.

In the case where there exist more than one syndrome response category of Type (i), *i.e.*, strictly increasing, the sum total of the upper asymptotes of their operating characteristics should not exceed unity, because of the constraint (3-1-2). When all the response categories to item g are syndrome response categories, this sum total should equal unity. A typical example may be the case where $U_{k_g}(\theta)$ is constant for all θ and $R_{k_g}(\theta)$ is strictly increasing in θ and satisfies the first formula of (3-2-1) for each syndrome response category of Type (i), under the constraint (3-1-2). In such a case we have

$$\text{(3-4-1)} \qquad \sum_{k_g*} [\lim_{\theta \to \infty} R_{k_g}(\theta)][1 - U_{k_g}(\theta)] \leq 1,$$

where $\sum_{k_g*}$ means the summation over all the categories of Type (i), and an equality holds when all the response categories to item g are syndrome response categories.

As was mentioned in Section 3.2, (3-2-1) is a sufficient condition for k_g to be a syndrome response category, but not a necessary condition, as far as we consider a particular pair of $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$. For illustrative purposes, we shall see an example, in which, (3-2-1) is not satisfied, and yet a syndrome response category is provided. Let $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ be such that

$$\text{(3-4-2)} \qquad \begin{cases} R_{k_g}(\theta) = (c_2 - c_1)(1 - c_1 + e^{-\theta})^{-1} \\ U_{k_g}(\theta) = c_1(1 + e^{-\theta})^{-1} \end{cases},$$

where c_1 and c_2 are constants, satisfying

$$\text{(3-4-3)} \qquad 0 < c_1 < c_2 < 1.$$

It can be proved that both $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ are strictly increasing in θ, with zero as their lower asymptotes and $(c_2 - c_1)(1 - c_1)^{-1}$ and c_1 as their respective upper asymptotes. Thus it is obvious that $[1 - U_{k_g}(\theta)]$ does not provide any one of the three types of functions given in Chapter 2, whereas $R_{k_g}(\theta)$ is of Type (i) and also satisfies the first formula of (3-2-1). Substituting (3-4-2) into (3-1-1) we obtain the operating characteristic of k_g such that

$$\text{(3-4-4)} \qquad P_{k_g}(\theta) = (c_2 - c_1)(1 + e^{-\theta})^{-1},$$

which proves that k_g is a syndrome response category of Type (i).

Similar discussion can be made for the case in which there exist more than one syndrome response category of Type (ii), *i.e.*, strictly decreasing. Situation B is not so likely to happen in typically nominal situations, however. It is more likely to happen in the heterogeneous case of the graded response level, which will be discussed in the following chapter.

CHAPTER 4

THE GRADED RESPONSE LEVEL (1)— THE HETEROGENEOUS CASE

On the graded response level, an item score is assigned to each item response category, in accordance with specified psychological criteria. This set of item score categories can either be enumerable or finite. Discussions will mainly be focused on the finite case, however, in this chapter and also in Chapter 5, since in practice we encounter mostly with the filnite case Thus in Sections, 4.1 through 4.5, it is assumed that the set of al the score. categories to item g is finite, whereas the enumerable case is discussed in Section 4.6.

4.1 Fundamental Concepts and Assumptions Particular to the Graded Response Level

As was introduced in Section 1.2, let x_g denote the item score or graded response category, as distinct from the nominal response category k_g , and, for convenience, let x_g be integers, 0 through m_g , which express the relative positions of the categories in their ordered sequence.

Since the graded response level is a subset or special case of the nominal level, fundamental discussions made in the preceding chapter are also valid in the present situation. The fundamental formula for the operating characteristic of the score category is given by replacing x_g for k_g in (3-1-1) such that

$$P_{x_g}(\theta) = R_{x_g}(\theta)[1 - U_{x_g}(\theta)]. \tag{4-1-1}$$

where $R_{x_g}(\theta)$ is the probability with which the subject of trait θ is attracted by the score category x_g , and $U_{x_g}(\theta)$ is the conditional probability that the rejection occurs to him, given that he has been attracted by x_g .

We shall assume on the graded level that a subject, who is attracted by the score category x_g , *automatically rejects* the category $(x_g - 1)$, by which he has been attracted. Thus if x_g is greater than 1, such a subject has a sequence of "being attracted by" and "the rejection of" previous categories. Although this assumption is rather schematic than descriptive in the sense that it does not necessarily indicate the temporal relationship of the subject's psychological reality, it may be acceptable in general.

Let $M_{x_g}(\theta)$ denote the conditional probability with which the subject of trait θ, who has already been attracted by the category $(x_g - 1)$, is further attracted by x_g . Following this logic, we can write

(4-1-2) $$R_{x_g}(\theta) = \prod_{s \le x_g} M_s(\theta)$$

and

(4-1-3) $$U_{x_g}(\theta) = M_{(x_g+1)}(\theta).$$

Using the symbol $P^*_{x_g}(\theta)$ instead of $R_{x_g}(\theta)$ and substituting (4-1-2) and (4-1-3) into (4-1-1), we obtain

(4-1-4) $$\begin{aligned} P_{x_g}(\theta) &= \prod_{s \le x_g} M_s(\theta)[1 - M_{(x_g+1)}(\theta)] \\ &= P^*_{x_g}(\theta) - P^*_{(x_g+1)}(\theta), \end{aligned}$$

which gives the fundamental formula for the operating characteristic of the score category in the graded level situation. Thus $M_s(\theta)$ for the s'S, 0 through $(x_g + 1)$, is the sole determinant of the operating characteristic, $P_{x_g}(\theta)$, and $P_{x_g}(\theta)$ is dependent upon the operating characteristics of preceding score categories.

Our second assumption on the graded level is that $M_{x_g}(\theta)$ is *either strictly increasing in θ or constant for all θ*. In particular, we assume that

(4-1-5) $$\begin{cases} M_0(\theta) = 1 \\ M_{(m_g+1)}(\theta) = 0 \end{cases},$$

and otherwise $M_{x_g}(\theta)$ is greater than zero and less than unity for all θ, when it is constant. Since in general n items are selected in such a way that each of them has some direct and positive significance to the trait measured, this assumption may be acceptable. Implications of (4-1-5) are obvious, considering the fact that there are neither lower categories than 0 nor higher categories than m_g . From (4-1-5) and the definition of $P^*_{x_g}(\theta)$ we have

(4-1-6) $$\begin{cases} P^*_0(\theta) = 1 \\ P^*_1(\theta) = M_1(\theta). \\ P^*_{(m_g+1)}(\theta) = 0 \end{cases}$$

We can conceive of a special case in which we have

(4-1-7) $$P^*_{x_g}(\theta) = M_1(\theta - \lambda_{x_g})$$

for the x_g's, 1 through m_g , where

(4-1-8) $$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_{m_g} < \infty.$$

It is obvious from (4-1-7) that in this case $P^*_{x_g}(\theta)$ for the x_g's, 1 through m_g , are identical, except for the positions on the trait continuum. We shall call this special situation the *homogeneous case* of the graded response level. Some characteristics of this case have been discussed by Samejima [Samejima, 1969, Chapter 4] and further observation will be made in Chapter 5.

As distinct from the homogeneous case, all the other cases will be categorized and called the *heterogeneous case*, or the term will be used for the general case of the graded response level, by which the homogeneous case is included as well. In the following sections of this chapter emphasis will be put upon the heterogeneous case as a special situation rather than the general situation of the graded response level.

Our third assumption on the graded level is that $M_{x_g}(\theta)$ is *three-times differentiable* with respect to θ. By virtue of (4-1-4) it follows that $P^*_{x_g}(\theta)$, and hence $P_{x_g}(\theta)$, are *also three-times differentiable*, which is consistent with the general assumption (3) in Section 1.2.

4.2 Sufficient Condition for the Score Categories to be Syndrome Response Categories

Since the operating characteristics of $(m_g + 1)$ score categories are not independent from one another, here we shall consider conditions with which all the $(m_g + 1)$ response categories to a specified item g are syndrome response categories, rather than those with respect to a single score category x_g .

The direct translation of (3-2-1) for the nominal level situation, which was discussed in the preceding chapter, gives

$$(4\text{-}2\text{-}1)\qquad \begin{cases} \sum_{s \le x_g} \frac{\partial^2}{\partial\theta^2} \log M_s(\theta) \le 0 \\ \frac{\partial^2}{\partial\theta^2} \log [1 - M_{(x_g+1)}(\theta)] \le 0 \end{cases}$$

where a strict inequality holds for all θ except, at most, for an enumerable number of points of θ, for the graded level situation as a sufficient, though not necessary, condition that the score category x_g should be a syndrome response category. Then its expanded form, which was discussed in Section 3.2, implies that an equality may hold for all θ in one of the formulas of (4-2-1) and a strict inequality should hold for all θ except, at most, for an an enumerable number of points of θ in the other. For category 0 this means

$$(4\text{-}2\text{-}2)\qquad \frac{\partial^2}{\partial\theta^2} \log [1 - M_1(\theta)] \le 0.$$

where a strict inequality holds for all θ except, at most, for an enumerable number of points, since by (4-1-5) it is obvious that the first formula of (4-2-1) has an equality for all θ. (4-2-2) is also a necessary condition as well as a sufficient condition, since we have from (4-1-4) and (4-1-6)

$$(4\text{-}2\text{-}3)\qquad P_0(\theta) = 1 - M_1(\theta),$$

and hence (4-2-2) is equivalent to (2-1), which gives the definition of the syndrome response category. This also implies that $M_1(\theta)$ should have unity as its upper asymptote. Given (4-2-2), the first formula of (4-2-1) means for the score category 1

$$\text{(4-2-4)} \qquad \frac{\partial^2}{\partial\theta^2} \log M_1(\theta) \leq 0,$$

where a strict inequality should hold for all θ except, at most, for an enumerable number of points. (4-2-4) implies that the lower asymptote of $M_1(\theta)$ should be zero, since $M_1(\theta)$ is a bounded function. Now it is interesting to note that in order for (4-2-1) to be true in its expanded form for the categories, 1 through m_g, $M_{x_g}(\theta)$ can be constant for all the other x_g's, 2 through m_g. In that case, an example of Situation B is provided, where the operating characteristic of category 0 is of Type (ii), *i.e.*, strictly decreasing in θ, and those for the categories, 1 through m_g, are of Type (i), *i.e.*, strictly increasing in θ. This is also a typical example of multi-correct responses discussed in Section 3.4. In order for (4-2-1) to provide Situation A, however, which has only one operating characteristic of Type (i), only one of Type (ii), and $(m_g - 1)$ of Type (iii), *i.e.*, uni-modal, $M_{x_g}(\theta)$ should not be constant for any of these categories. In that case, it should be strictly increasing in θ with unity as its asymptote for the x_g's, 2 through m_g, whereas its lower asymptote does not necessarily have to be zero. A typical example of such a case will be given in Section 5.2 of the next chapter, when the logistic model of the homogeneous case is discussed.

The fact that (4-2-1) is not a necessary condition for x_g to be a syndrome score category, even in its expanded form, can easily be seen from the fact that it does not provide two or more score categories whose operating characteristics are of Type (iii) although this is permitted in Situation B. In that case, $M_1(\theta)$ should have some positive value as its lower asymptote, which is not provided by (4-2-1).

In any case, under the present assumption, category 0 can only be a syndrome response category of Type (ii). On the other hand, since we have from (4-1-4) and the second formula of (4-1-5)

$$\text{(4-2-5)} \qquad P_{m_g}(\theta) = \prod_{s \leq m_g} M_s(\theta)$$

for category m_g, it can only be a syndrome response category of Type (i). More diversity is allowed for the categories, 1 through $(m_g - 1)$, which will be observed in more detail in the following section.

Suppose that the formulas for $M_{x_g}(\theta)$ for the categories, 1 through m_g, are independent from one another. Suppose, further, we consider a family of formulas expanded from each $M_{x_g}(\theta)$ in such a way that

$$\text{(4-2-6)} \qquad M^*_{x_g}(\theta) = M_{x_g}(\alpha_{x_g}\theta + \beta_{x_g})$$

where α_{x_g} is a positive constant and β_{x_g} is a real constant. It is easily seen that the necessary and sufficient condition that *any* combination of members of these families should provide a set of $(m_g + 1)$ syndrome score categories in Situation A, and, moreover, in so doing $M^*_{x_g}(\theta)$ can be used interchangeably among categories, is the satisfaction of:

$$\begin{cases} \dfrac{\partial^2}{\partial\theta^2} \log M_{x_g}(\theta) \leq 0 \\ \dfrac{\partial^2}{\partial\theta^2} \log [1 - M_{x_g}(\theta)] \leq 0 \end{cases} \tag{4-2-7}$$

for each and every $M_{x_g}(\theta)$, where a strict inequality should hold for all θ except, at most, for an enumerable number of points in each formula. Such a $M_{x_g}(\theta)$ is given by $W(\theta)$ itself, which was introduced in the preceding chapter, *i.e.*, any three-times-differentiable and strictly increasing function of θ with zero and unity as its lower and upper asymptotes, which satisfies

$$\frac{\partial^2}{\partial\theta^2} \log \left[\frac{\partial}{\partial\theta} W(\theta)\right] \leq 0, \tag{3-2-3}$$

where an equality holds, at most, at an enumerable number of points. It should be reminded that the first derivative of $W(\theta)$ should not equal zero at any point of θ in order to satisfy (3-2-3).

4.3 Orderliness and Reclassification of Syndrome Score Categories

Suppose we have an item g the score categories of which are syndrome response categories. Do the operating characteristics of these score categories reflect the rank order in some way or another? In fact, the answer is negative in many examples, though positive in others, and this is one of the characteristics of the heterogeneous case as distinct from the homogeneous case.

As a measure of orderliness of the score categories of a single item, we could take the modal points of their operating characteristics. As was mentioned in Section 3.2, in Situation B there are more than one score category whose operating characteristic is of Type (i), *i.e.*, strictly increasing, or more than one whose operating characteristic is of Type (ii), *i.e.*, strictly decreasing, or both. This fact suggests there are two or more score categories which uniformly have negative or positive terminal maxima, and among them there is no order, even though their scores are ordered. In fact, Situation B includes the case in which none of the operating characteristics of the score categories are of Type (iii), *i.e.*, uni-modal, but all of them are of Types (i) or (ii), even though m_g is a large number. It is easily seen that, in order to have more than one syndrome response category of Type (ii), the lower asymptote of $M_1(\theta)$ should be greater than zero, which evidently does not satisfy the first formula of (4-2-1), as was observed earlier. Thus categories 0 and 1 should necessarily be of Type (ii) in this case. If $M_2(\theta)$ is also strictly increasing in θ with the lower asymptote greater than zero, it possibly provides a syndrome response category of Type (ii) for category 2. In this way, we can conceive of at most m_g categories of Type (ii) having negative infinity as their common terminal maximum, with the sole exception of the highest score category, m_g , being of Type (i). Similarly, we can think of a case in which there are m_g score categories of Type (i) having positive infinity as their common terminal maximum, with the exception of category 0 which has negative infinity as its terminal maximum. Besides, we can even conceive of a case where categories of lower scores have *positive* infinity as their terminal maxima whereas those of higher scores have some local maxima, and also a case where categories of higher scores have *negative* infinity as their terminal maxima while those of lower scores have local maxima, although categories 0 and m_g always have negative and positive terminal maxima respectively and no two categories can be possessed of positive and negative infinities as their terminal maxima in the reversed order of their scores.

These examples indicate the *disorderliness* of the modal points of the operating characteristics of score categories, which suggests that in the heterogeneous case we must strictly distinguish the order of the modal points from the rank order of scores attached to the response categories.

In Situation A where categories 0 and m_g are sole categories which have negative and positive infinities as their terminal maxima respectively, we could expect more orderliness than in Situation B. Extending the observation made on Situation B, however, we can see that, in the general case, the complete orderliness is not reached in Situation A either. To explain this, suppose $\prod_{s \leq x_g} M_s(\theta)$ is a strictly increasing function of θ with zero and unity as its lower and upper asymptotes. Then we can conceive of $M_{(x_g+1)}(\theta)$ which provides a syndrome response category for x_g and yet as close to a constant as we wish. In such a case, the modal point of category x_g can be as high as we wish, to exceed the local maximum of the next category, $(x_g + 1)$, if an appropriate $M_{(x_g+2)}(\theta)$ is given.

For the purpose of illustration, we shall consider an example, in which

$$\begin{cases} P_1^*(\theta) = \Phi(-0.4, 1) \\ P_2^*(\theta) = \Phi(0.0, 1) \\ M_3(\theta) = \Phi(-1.2, 0.01) \end{cases} \tag{4-3-1}$$

where Φ indicates the normal ogive function, and the two numbers in the brackets indicate its mean and variance respectively. These functions provide syndrome response categories for both $x_g = 1$ and $x_g = 2$, as we shall see in Sections 5.2 and 5.4. Figure 4-3-1 presents $P_1^*(\theta)$, $P_2^*(\theta)$, and $P_3^*(\theta)$ by dashed line, and $M_2(\theta)$ and $M_3(\theta)$ by solid line. We can see that approximately 85% of the upper part of $P_3^*(\theta)$ almost overlaps that of $P_2^*(\theta)$. Figure 4-3-2

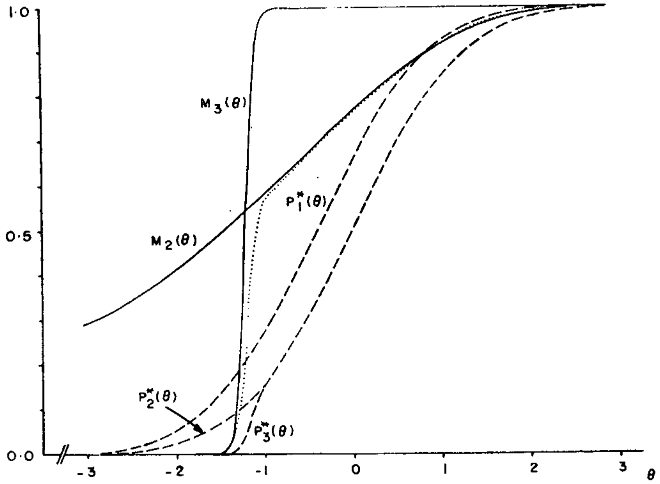

FIGURE 4-3-1

$P_1^*(\theta)$, $P_2^*(\theta)$ and $P_3^*(\theta)$ (dashed line) and $M_2(\theta)$ and $M_3(\theta)$ (solid line) of example 1, in which $P_1^*(\theta)$, $P_2^*(\theta)$ and $M_3(\theta)$ are given by (4-3-1). Also $M_{x_g}(\theta)$ of the new category resulted by combining the categories 1 and 2 is presented (dotted line).

presents $P_1(\theta)$, *i.e.*, the operating characteristic of category 1, by dotted line, and $P_2(\theta)$, *i.e.*, that of category 2, by dashed line. We can see in this figure that not only the local maximum for category 1 is far greater than the local maximum for category 2, but the relative positions of these two curves seem to be in the reversed order of the scores. This rather peculiar result comes from the fact that $M_3(\theta)$ has a very steep curve compared with $M_2(\theta)$, as we can see in Figure 4-3-1. In other words, passing through the boundary between categories 1 and 2 does not require a psychological process which is closely related to the trait, while passing through the boundary between

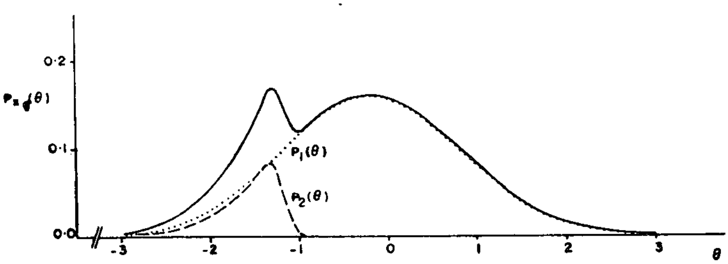

FIGURE 4-3-2

$P_1(\theta)$ (dotted line) and $P_2(\theta)$ (dashed line), *i.e.*, the operating characteristics of categories 1 and 2, of example 1. Also the operating characteristic of the new category resulted by combining the categories 1 and 2 is presented (solid line).

categories 2 and 3 has a very high discriminating power around $\theta = -1.2$, and, as a result, $P_2(\theta)$ has a sharper curve than $P_1(\theta)$.

It is easily seen that a sufficient, though not necessary, condition that the complete orderliness of modal points for the syndrome score categories should be reached in the order of the category scores is the satisfaction of

$$(4\text{-}3\text{-}2) \qquad A_{(x_g-1)}(\theta) < A_{x_g}(\theta)$$

for all θ for the x_g's, 1 through m_g , where $A_{x_g}(\theta)$ is the basic function of category x_g . From (4-1-4) and (1-2-5), the definition of the basic function, we can rewrite (4-3-2) into the form

$$(4\text{-}3\text{-}3) \qquad \frac{\frac{\partial}{\partial\theta} M_{x_g}(\theta)}{M_{x_g}(\theta)[1 - M_{x_g}(\theta)]} > \frac{\frac{\partial}{\partial\theta} M_{(x_g+1)}(\theta)}{1 - M_{(x_g+1)}(\theta)}.$$

This leads to the fact that if we have

$$(4\text{-}3\text{-}4) \qquad \frac{\partial}{\partial\theta} \log [1 - M_{x_g}(\theta)] \leq \frac{\partial}{\partial\theta} \log [1 - M_{(x_g+1)}(\theta)]$$

for the x_g's, 1 through m_g , the resulting operating characteristics provide strictly ordered modal points for $(m_g + 1)$ syndrome score categories.

For the purpose of illustration, we shall consider another example, in which (4-3-4) is satisfied. Let $m_g = 4$, and $M_{x_g}(\theta)$ is given by

$$(4\text{-}3\text{-}5) \qquad M_{x_g}(\theta) = 1 - \exp \{-e^{(\theta - b_{x_g})}\}$$

for the x_g's, 1 through 4, where $b_{x_g} = 0.0, 0.5, 1.0, 1.5$ respectively. Since we have from (4-3-5)

$$(4\text{-}3\text{-}6) \qquad \frac{\partial^2}{\partial\theta^2} \log \left[\frac{\partial}{\partial\theta} M_{x_g}(\theta)\right] = -e^{(\theta - b_{x_g})} < 0$$

and it is obvious that the lower and upper asymptotes of $M_{x_g}(\theta)$ are zero and unity respectively, (4-3-5) provides Situation A, as was seen in Section 4.2. Further, we obtain

$$(4\text{-}3\text{-}7) \qquad \frac{\partial}{\partial\theta} \log [1 - M_{x_g}(\theta)] = -e^{(\theta - b_{x_g})}$$

which leads to the satisfaction of (4-3-4) with a strict inequality. Thus all the score categories of this example are syndrome response categories and, moreover, the complete orderliness of their modal points is reached. Figure 4-3-3 presents the five operating characteristics of this example by solid line, $P^*_{x_g}(\theta)$ for the x_g's, 1 through 3, by dashed line, and $M_{x_g}(\theta)$ for the x_g's, 2 through 4, by dotted line. Note that $P^*_4(\theta) = P_4(\theta)$, and $M_1(\theta) = P^*_1(\theta)$.

Now we shall proceed to discuss the *reclassification* of syndrome response categories. If we combine two adjacent syndrome response categories, the

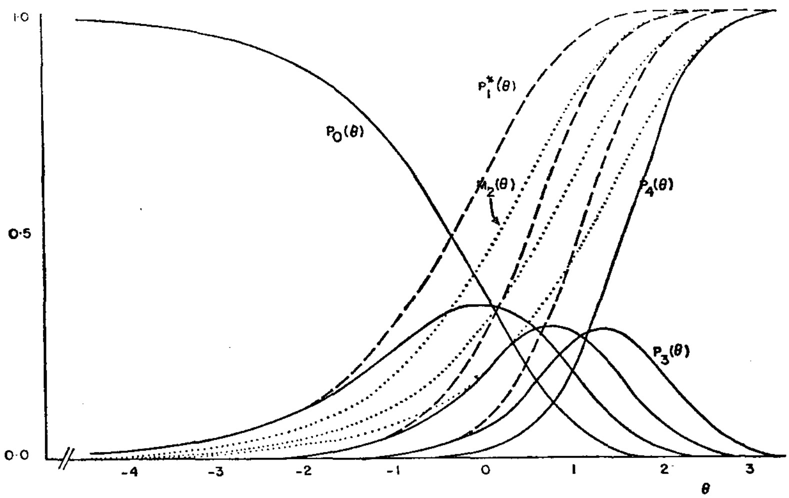

FIGURE 4–3–3

$P_{x_g}(\theta)$ for the categories, 0 through 4, *i.e.*, the operating characteristics of these categories (solid line), $P^*_{x_g}(\theta)$ for the categories, 1 through 3, (dashed line), and $M_{x_g}(\theta)$ for the categories, 2 through 4, (dotted line), of example 2, in which $M_{x_g}(\theta)$ is given by (4-3-5) where $b_{x_g} = 0.0, 0.5, 1.0, 1.5$ for the x_g's, 1 through 4 respectively.

resulting new category is not always a syndrome response category. Figure 4-3-2 also presents the operating characteristic of the new category, *i.e.*, the combination of categories 1 and 2, by solid line, which shows a conspicuous bi-modality. This new category is obviously not a syndrome response category, though $M_{x_g}(\theta)$ for the new category, which is the product of $M_2(\theta)$ and $M_3(\theta)$, is a strictly increasing function of θ with zero and unity as its lower and upper asymptotes, as is shown by dotted line in Figure 4-3-1. This is a typical example of the case in which reclassification of syndrome response categories does not produce a syndrome response category, although even in the heterogeneous case it can provide a new syndrome score category depending upon the functional formula of the resulting $M_{x_g}(\theta)$.

Thus we have seen in this section that in the heterogeneous case the scores assigned to syndrome response categories do not necessarily reflect the order of their modal points or maximum likelihood estimates, and also a new category obtained by combining two adjacent syndrome response categories is not always a syndrome response category. If we consider Situation B in addition to Situation A, these observations become more impressive. This fact might lead us to question whether the heterogeneous case has a psychological significance. For this reason, we shall consider a general psychological process which can be explained by the heterogeneous case of the graded response level.

Suppose a specified psychological process can be categorized into a finite number of steps. Let x_g denote a step or graded category, and assume that, in order to proceed to the step x_g , the subject *must* have passed all the preceding steps. Thus each boundary between steps is a *discriminating process*, in which $M_{x_g}(\theta)$ takes an important role. If $M_{x_g}(\theta)$ is steep, it is highly discriminating, and if it is even, its discrimination power is low, and the resulting likelihood function, *i.e.*, operating characteristic, is affected, as we have seen in Figures 4-3-1 and 4-3-2. In an extreme case where a discriminating process has no correlation with the trait, *i.e.*, $M_{x_g}(\theta)$ is constant for all θ, or where its slope is very little, category x_g possibly has a terminal maximum at positive infinity, even if it is one of the middle categories.

At any rate, in measuring a psychological trait, it may not be desirable to use an item involving a little discriminating process. In this sense, the orderliness of the modal points for syndrome score categories can be a measure of adequacy of the item, although their kurtosis may be important as well.

4.4 Bock's Multinomial Response Model as an Example of the Heterogeneous Case

A multinomial model has been suggested by Bock by generalizing the logistic response law and using the multivariate logits [Bock, 1966]. In this model, the operating characteristic of category k under the experimental condition j is given by

$$P_{jk} = \frac{e^{z_{jk}}}{\sum_{s=1}^{m} e^{z_{js}}}, \tag{4-4-1}$$

where z_{jk} is a multivariate logit, and m (≥ 2) is the number of response categories.

If we apply this model for the uni-variate situation and change the notation into the present one, we can rewrite (4-4-1) into the form

$$P_{k_g}(\theta) = \frac{e^{\alpha_{k_g}\theta + \beta_{k_g}}}{\sum_{s} e^{\alpha_s\theta + \beta_s}} \tag{4-4-2}$$

where α's are positive constants and β's are real constants. Although originally this was presented as a nominal model by Bock, in the present system, it is a typical example of the heterogeneous case of the graded response level.

Suppose that these response categories are scored in accordance with the magnitude of α. Thus we have

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{m_g} . \tag{4-4-3}$$

Let $M_{x_g}(\theta)$ be such that

$$(4\text{-}4\text{-}4) \qquad M_{x_g}(\theta) = \frac{\sum_{s=x_g}^{m_g} e^{\alpha_s\theta+\beta_s}}{\sum_{s=x_g-1}^{m_g} e^{\alpha_s\theta+\beta_s}}$$

for the x_g's, 1 through m_g. Differentiating (4-4-4) with respect to θ, we have

$$(4\text{-}4\text{-}5) \qquad \frac{\partial}{\partial\theta} M_{x_g}(\theta) = \frac{\sum_{s=x_g}^{m_g} e^{(\alpha_s+\alpha_{(x_g-1)})\theta}(\alpha_s - \alpha_{(x_g-1)})e^{\beta_s+\beta_{(x_g-1)}}}{\left[\sum_{s=x_g-1}^{m_g} e^{\alpha_s\theta+\beta_s}\right]^2}$$

which equals zero if, and only if, the α_s's for categories, $(x_g - 1)$ through m_g, take the same value, and is greater than zero if, and only if, at least, one of them is different from another. This fact indicates that $M_{x_g}(\theta)$ is either strictly increasing in θ or constant for all θ, which satisfies the fundamental assumption made for $M_{x_g}(\theta)$ in Section 4.1. Since we can rewrite (4-4-4) into the form

$$(4\text{-}4\text{-}6) \qquad M_{x_g}(\theta) = \frac{1}{1 + \frac{1}{\sum_{s=x_g}^{m_g} e^{(\alpha_s-\alpha_{(x_g-1)})\theta}e^{\beta_s-\beta_{(x_g-1)}}}},$$

we have

$$(4\text{-}4\text{-}7) \qquad M_{x_g}(\theta) = \frac{\sum_{s=x_g}^{m_g} e^{\beta_s}}{e^{\beta_{(x_g-1)}} + \sum_{s=x_g}^{m_g} e^{\beta_s}}$$

when $M_{x_g}(\theta)$ is constant, and

$$(4\text{-}4\text{-}8) \qquad \begin{cases} \lim_{\theta\to-\infty} M_{x_g}(\theta) \geq 0 \\ \lim_{\theta\to\infty} M_{x_g}(\theta) = 1 \end{cases}$$

when it is strictly increasing in θ.

From (4-1-2) and (4-4-4) we can write

$$(4\text{-}4\text{-}9) \qquad P^*_{x_g}(\theta) = \prod_{s\leq x_g} M_s(\theta) = \frac{\sum_{s=x_g}^{m_g} e^{\alpha_s\theta+\beta_s}}{\sum_{s=0}^{m_g} e^{\alpha_s\theta+\beta_s}},$$

and inserting this into (4-1-4) we obtain

$$P_{x_g}(\theta) = \frac{e^{\alpha_{x_g}\theta + \beta_{x_g}}}{\sum_{s=0}^{m_g} e^{\alpha_s\theta + \beta_s}} \tag{4-4-10}$$

which is identical with (4-4-2). Thus it has been shown that, defining $M_{x_g}(\theta)$ by (4-4-4), we can specify the Bock model as an example of the heterogeneous case of the graded response level, although it was originally developed for the non-ordered case.

Now the question is whether this model provides syndrome score categories or not. If the α_s's for all categories, 0 through m_g, are the same value, from (4-4-10) we have

$$P_{x_g}(\theta) = \frac{e^{\beta_{x_g}}}{\sum_{s=0}^{m_g} e^{\beta_s}}, \tag{4-4-11}$$

which indicates that all the operating characteristics are constant for all θ, and, therefore, none of the categories is a syndrome response category.

Let us suppose, then, that, at least, one of the α_s's has a different value from another. Differentiating (4-4-10) with respect to θ, we have

$$\frac{\partial}{\partial\theta} P_{x_g}(\theta) = \frac{\sum_{s=0}^{m_g} (\alpha_{x_g} - \alpha_s) e^{\alpha_s\theta + \beta_s}}{\sum_{s=0}^{m_g} e^{\alpha_s\theta + \beta_s}} P_{x_g}(\theta) \tag{4-4-12}$$

and we can write for the basic function from (4-4-10) and (4-4-12)

$$A_{x_g}(\theta) = \frac{\frac{\partial}{\partial\theta} P_{x_g}(\theta)}{P_{x_g}(\theta)} = \frac{\sum_{s=0}^{m_g} (\alpha_{x_g} - \alpha_s) e^{\alpha_s\theta + \beta_s}}{\sum_{s=0}^{m_g} e^{\alpha_s\theta + \beta_s}}. \tag{4-4-13}$$

Since we have

$$\begin{aligned} &\left\{\sum_{s=0}^{m_g} e^{\alpha_s\theta + \beta_s}\right\}\left\{\sum_{s=0}^{m_g} \alpha_s(\alpha_{x_g} - \alpha_s) e^{\alpha_s\theta + \beta_s}\right\} \\ &\quad - \left\{\sum_{s=0}^{m_g} (\alpha_{x_g} - \alpha_s) e^{\alpha_s\theta + \beta_s}\right\}\left\{\sum_{s=0}^{m_g} \alpha_s e^{\alpha_s\theta + \beta_s}\right\} \\ &= \sum_{s=0}^{m_g} \sum_{t=0}^{m_g} \alpha_t(\alpha_s - \alpha_t) e^{(\alpha_s + \alpha_t)\theta} e^{\beta_s + \beta_t} \end{aligned} \tag{4-4-14}$$

$$= \sum_{s=0}^{m_g} \sum_{t>s} [-(\alpha_s - \alpha_t)^2 e^{(\alpha_s+\alpha_t)\theta} e^{\beta_s+\beta_t}]$$
$$< 0,$$

we obtain

$$\frac{\partial}{\partial\theta} A_{x_g}(\theta) < 0, \tag{4-4-15}$$

which means that x_g is a syndrome response category. Thus we have seen that the Bock model provides syndrome score categories for all the x_g's under the restriction that, at least, one of the α_s's is different from another.

Setting (4-4-13) equal to zero, we obtain

$$\alpha_{x_g} = \frac{\sum_{s=0}^{m_g} \alpha_s e^{\alpha_s\theta+\beta_s}}{\sum_{s=0}^{m_g} e^{\alpha_s\theta+\beta_s}}, \tag{4-4-16}$$

so that the value of θ satisfying (4-4-16) is the modal point for category x_g. Since the right hand side of (4-4-16) is the weighted sum of the α_s's and thus strictly increasing in θ, we can see that, if, and only if, all the α_s's take different values from one another, *i.e.*, if, and only if, strict inequalities hold in (4-4-3), a perfect orderliness is reached, with terminal maxima at negative and positive infinities for $x_g = 0$ and $x_g = m_g$ respectively, and unique local maxima otherwise. In other words, in such a case, a perfect orderliness is realized in Situation A, with one operating characteristic of type (i), another of type (ii), and all the other $(m_g - 1)$ of type (iii), satisfying

$$\begin{cases} \lim_{\theta\to-\infty} P_{x_g}(\theta) = 1; & \text{for} \quad x_g = 0 \\ \lim_{\theta\to-\infty} P_{x_g}(\theta) = 0; & \text{otherwise} \end{cases} \tag{4-4-17}$$

and

$$\begin{cases} \lim_{\theta\to\infty} P_{x_g}(\theta) = 1; & \text{for} \quad x_g = m_g \\ \lim_{\theta\to\infty} P_{x_g}(\theta) = 0; & \text{otherwise} \end{cases} . \tag{4-4-18}$$

The formula (4-4-16) also suggests that, if there are more than one category which share the least value of α_s, the operating characteristics of these categories are uniformly of type (ii), *i.e.*, strictly decreasing in θ, and, similarly, if there are more than one category which share the greatest value of α_s, the operating characteristics of all these categories are of type (i), *i.e.*, strictly increasing in θ. In other words, if one of them, or both, takes place, Situation B is provided. Since we can rewrite (4-4-10) into the form

$$P_{x_g}(\theta) = \frac{1}{\sum_{s=0}^{m_g} e^{(\alpha_s - \alpha_{x_g})\theta} e^{\beta_s - \beta_{x_g}}}, \tag{4-4-19}$$

in the former case we have for the upper asymptotes of these strictly decreasing operating characteristics

$$\lim_{\theta \to -\infty} P_{x_g}(\theta) = \frac{1}{\sum_{s^*} e^{\beta_s - \beta_{x_g}}}, \tag{4-4-20}$$

where $\sum_{s^*}$ means the summation over all the categories sharing the common least value of α_s, and in the latter case we have for the upper asymptotes of these strictly increasing ones

$$\lim_{\theta \to \infty} P_{x_g}(\theta) = \frac{1}{\sum_{s^{**}} e^{\beta_s - \beta_{x_g}}}, \tag{4-4-21}$$

where $\sum_{s^{**}}$ indicates the summation over all the categories sharing the common greatest value of α_s.

We can also see from (4-4-16) that, if more than one category share a common value of α_s which is neither least nor greatest, they also share a common local maximum. If these categories also share a common value of β_{x_g}, their operating characteristics are identical uni-modal functions of θ, as is obvious from (4-4-10). The same is true in earlier two cases, in which identical strictly monotone decreasing and increasing functions are shared respectively. In the second case, $M_{(x_g+1)}(\theta)$ is constant for all θ with respect to these categories, as is obvious from (4-4-6).

Thus we have seen that Bock's multinomial response model can be regarded as a typical example of the heterogeneous case of the graded response model, which may provide either Situation A or Situation B, although originally it was presented as a model for the non-ordered case.

4.5 Necessary and Sufficient Condition that $M_{x_g}(\theta)$ should be Strictly Increasing in θ

So far discussions have been made on the assumption that $M_{x_g}(\theta)$ is either constant or strictly increasing in θ, for all θ. In practical situations, however, it is also likely to happen that a set of $P^*_{x_g}(\theta)$ is given for the categories, 1 through m_g, and a question is whether for these categories $M_{x_g}(\theta)$ is a strictly increasing function of θ or not. In this section, therefore, we shall see the necessary and sufficient condition that $M_{x_g}(\theta)$ should be strictly increasing in θ, given a set of $P^*_{x_g}(\theta)$.

We can write from (4-1-2) that

(4-5-1) $$M_{x_g}(\theta) = P^*_{x_g}(\theta)/P^*_{(x_g-1)}(\theta)$$

for the x_g's, 1 through m_g . Differentiating this with respect to θ, we have

(4-5-2) $$\frac{\partial}{\partial\theta} M_{x_g}(\theta) = \frac{P^*_{(x_g-1)}(\theta) \frac{\partial}{\partial\theta} P^*_{x_g}(\theta) - P^*_{x_g}(\theta) \frac{\partial}{\partial\theta} P^*_{(x_g-1)}(\theta)}{P^*_{(x_g-1)}(\theta)^2}$$

which should be positive, or equal to zero at most at an enumerable number of points, in order for $M_{x_g}(\theta)$ to be strictly increasing in θ. Since the denominator of (4-5-2) is positive for all θ, the above statement should be realized in its numerator.

It is easily seen that this is true, if, and only if, we have

(4-5-3) $$\frac{\partial}{\partial\theta} \log P^*_{(x_g-1)}(\theta) \leq \frac{\partial}{\partial\theta} \log P^*_{x_g}(\theta)$$

for the x_g's, 1 through m_g , throughout the whole range of θ, where an equality holds at most at an enumerable number of points. For we can write

(4-5-4) $$\frac{\partial}{\partial\theta} \log P^*_{x_g}(\theta) - \frac{\partial}{\partial\theta} \log P^*_{(x_g-1)}(\theta)$$
$$= \frac{P^*_{(x_g-1)}(\theta) \frac{\partial}{\partial\theta} P^*_{x_g}(\theta) - P^*_{x_g}(\theta) \frac{\partial}{\partial\theta} P^*_{(x_g-1)}(\theta)}{P^*_{x_g}(\theta) P^*_{(x_g-1)}(\theta)}$$

which has a positive denominator for any value of θ, and whose numerator is exactly the same as that of (4-5-2). Thus a question whether $M_{x_g}(\theta)$ for a specified category is strictly increasing in θ or not is completely detectable if $P^*_{x_g}(\theta)$ and $P^*_{(x_g-1)}(\theta)$ are given. This is true in the heterogeneous case as well as in the homogeneous case.

4.6 Enumerable Set of Syndrome Score Categories

In previous four sections of the graded response level, we have assumed that the set of item scores or graded response categories is finite, 0 through $m_g(< \infty)$. We can conceive of the situation, however, in which it is enumerable, although at this moment the author is not certain about its practical importance. In this section, we shall add some discussion about the enumerable set of graded item response categories.

The fundamental formula for the operating characteristic, (4-1-4), is also valid in the enumerable case. If the first or last category, or both, exists, either one of the formulas (4-1-5), or both, is also reasonably assumed, and hence we obtain the first two formulas or the last one of (4-1-6), or both, although m_g is not an adequate notation in these formulas. The homogeneous case is also defined by (4-1-7), by replacing M_1 by P^*_r , where r is an arbitrary category neither first nor last, and defining λ_{x_g} in such a way that

$$\text{(4-6-1)} \qquad \lambda_{x_g} \begin{cases} >0; & \text{for categories less than } r \\ =0; & \text{for category } r \\ <0; & \text{for categories greater than } r, \end{cases}$$

by keeping their relative magnitudes suggested by (4-1-8).

Suppose that all the score categories to item g are syndrome response categories in the enumerable situation. In the *homogeneous case,* it is obvious from the discussions made in previous sections that either Situation A or Situation C occurs. In other words, there are four possible situations, each including an enumerable number of uni-modal operating characteristics and the following in addition.

(1) One strictly decreasing operating characteristic, and one strictly increasing operating characteristic.
(2) One strictly decreasing operating characteristic.
(3) One strictly increasing operating characteristic.
(4) None.

Since the orderliness of the modal points for the homogeneous case, which will be discussed in Section 5.3, is also valid for the enumerable situation, in these four cases the modal points of the infinite number of categories are arranged strictly in the order of scores assigned to the categories, including one terminal maximum at negative infinity in (2), one terminal maximum at positive infinity in (3), and both in (1).

In the *heterogeneous case,* the orderliness of the modal points of syndrome response categories is not necessarily reached, as was observed in the finite situation. We can even conceive of examples, in which there exist an infinite number of non-ordered syndrome response categories. To give one, suppose that the first formula of (4-2-1) is satisfied for category r, which contains an infinite number of $M_{x_g}(\theta)$ whose item scores are no greater than r, and $P_r^*(\theta)$ is strictly increasing in θ with zero and unity as its lower and upper asymptotes. Suppose, further, that

$$\text{(4-6-2)} \qquad M_{x_g}(\theta) = \tfrac{1}{2}, \quad \text{for} \quad x_g = r + 1, r + 2, r + 3, \cdots$$

Thus (4-2-1) is fulfilled in its expanded form for all the categories greater than or equal to r, and all of them are syndrome response categories, with operating characteristics of type (i), *i.e.*, strictly increasing, having upper asymptotes given by

$$\text{(4-6-3)} \qquad \lim_{\theta \to \infty} P_{x_g}(\theta) = 2^{r - x_g - 1}.$$

There are an infinite number of such categories, and each of them has a terminal maximum at positive infinity. In a more extreme example, there may be an infinite number of categories, each of which has a terminal maximum at negative infinity, and an infinite number of categories which have a terminal maximum at positive infinity each, in addition to an infinite number of categories, which have a uni-modal operating characteristic each and whose modal points are not necessarily arranged in the order of scores attached to them.

CHAPTER 5

THE GRADED RESPONSE LEVEL (2)—THE HOMOGENEOUS CASE

Fundamental concepts and assumptions were given in Chapters 1 and 2, and those particular to the graded response level were given in Chapter 4, and discussions will be based on these concepts and assumptions in this chapter.

As we have seen in Section 4.1, the homogeneous case is characterized by the formula

$$(4\text{-}1\text{-}7) \qquad P^*_{x_g}(\theta) = M_1(\theta - \lambda_{x_g})$$

for the x_g's, 1 through m_g, where

$$(4\text{-}1\text{-}8) \qquad 0 = \lambda_1 < \lambda_2 < \cdots\cdots < \lambda_{m_g} < \infty .$$

An additional assumption particular to the homogeneous case is that $M_1(\theta)$ is *strictly increasing* in θ. Since $P_{x_g}(\theta)$ equals zero for all θ for the categories, 1 through $(m_g - 1)$, if $M_1(\theta)$ is constant, as is obvious from (4-1-4) and (4-1-7), this assumption may reasonably be acceptable. From this assumption and (4-1-7) $P^*_{x_g}(\theta)$ is also strictly increasing in θ for the x_g's, 1 through m_g.

5.1 Asymptotic Basic Function

It has been demonstrated by Samejima [Samejima, 1969, pages 31–33], that in the homogeneous case of the graded response level the asymptotic formula of the basic function, $A_{x_g}(\theta)$, when $\lambda_{(x_g+1)}$ tends to λ_{x_g} is given by

$$(5\text{-}1\text{-}1) \qquad \lim_{\lambda_{(x_g+1)} \to \lambda_{x_g}} A_{x_g}(\theta) = \frac{\partial^2}{\partial\theta^2} P^*_{x_g}(\theta) \Big/ \frac{\partial}{\partial\theta} P^*_{x_g}(\theta)$$

for the x_g's, 1 through $(m_g - 1)$, provided that

$$(5\text{-}1\text{-}2) \qquad \frac{\partial}{\partial\theta} P^*_{x_g}(\theta) > 0$$

for all θ. In fact, if the first derivative of $P^*_{x_g}(\theta)$ equals zero at a certain point of θ, then it takes a local *minimum*, which means that the second derivative of $P^*_{x_g}(\theta)$ is also zero at that point, and consequently the right hand side of (5-1-1) is 0/0. We can easily see from (4-1-7) that (5-1-2) is equivalent to

$$(5\text{-}1\text{-}3) \qquad \frac{\partial}{\partial\theta} M_1(\theta) > 0.$$

The right hand side of (5-1-1) can also be defined for $x_g = m_g$, and hereafter we shall call this term defined for each x_g, 1 through m_g, the *asymptotic basic function* of category x_g , and denote it by $\tilde{A}_{x_g}(\theta)$. Thus we can write from (4-1-7)

$$(5\text{-}1\text{-}4) \qquad \tilde{A}_{x_g}(\theta) = \frac{\partial^2}{\partial\theta^2} P^*_{x_g}(\theta) \Big/ \frac{\partial}{\partial\theta} P^*_{x_g}(\theta) = \frac{\partial^2}{\partial\theta^2} M_1(\theta - \lambda_{x_g}) \Big/ \frac{\partial}{\partial\theta} M_1(\theta - \lambda_{x_g}),$$

provided that $M_1(\theta)$ satisfies (5-1-3). Thus, in order for the m_g score categories to have their asymptotic basic functions, not only $M_1(\theta)$ should be strictly increasing in θ but also its first derivative should not equal zero at any point of θ. Although the homogeneous case is defined without this restriction, the asymptotic basic function is defined only for the case where (5-1-2), or (5-1-3), is true.

The sign of the asymptotic basic function is determined by the sign of the second derivative of $M_1(\theta - \lambda_{x_g})$, by virtue of (5-1-3). Since $M_1(\theta)$ is a bounded function and three-times differentiable, its second derivative necessarily has both positive and negative values for certain intervals of θ respectively, and takes zero at least at one point of θ, and so does the asymptotic basic function.

It is easily seen from (5-1-4) that the asymptotic basic function has an identical curve for the categories, 1 through m_g , except for the positions on the trait continuum. This leads to the fact that all of them have the same limits when θ tends to negative and positive infinities respectively. It is worth noting that the two limits of the asymptotic basic function coincide with the limits of the basic functions for the categories, 1 through $(m_g - 1)$. To prove this, by Cauchy's mean value theorem we can write

$$(5\text{-}1\text{-}5) \qquad A_{x_g}(\theta) = \left[\frac{\dfrac{\partial^2}{\partial\theta^2} M_1(\theta)}{\dfrac{\partial}{\partial\theta} M_1(\theta)}\right]_{\theta=\zeta} = \tilde{A}_1(\zeta)$$

for any category, 1 through $(m_g - 1)$, where ζ is some value of θ which satisfies

$$(5\text{-}1\text{-}6) \qquad \theta - \lambda_{(x_g+1)} < \zeta < \theta - \lambda_{x_g} .$$

It is easily seen that ζ tends to negative infinity as θ tends to negative infinity, and to positive infinity as θ tends to positive infinity. Thus we obtain

$$(5\text{-}1\text{-}7) \qquad \begin{cases} \lim\limits_{\theta\to-\infty} \tilde{A}_1(\theta) = \lim\limits_{\theta\to-\infty} \tilde{A}_2(\theta) = \cdots = \lim\limits_{\theta\to-\infty} \tilde{A}_{m_g}(\theta) \\ \qquad = C_{1,\underline{\theta}} = C_{2,\underline{\theta}} = \cdots = C_{(m_g-1),\underline{\theta}} \\ \lim\limits_{\theta\to\infty} \tilde{A}_1(\theta) = \lim\limits_{\theta\to\infty} \tilde{A}_2(\theta) = \cdots = \lim\limits_{\theta\to\infty} \tilde{A}_{m_g}(\theta) \\ \qquad = C_{1,\bar{\theta}} = C_{2,\bar{\theta}} = \cdots = C_{(m_g-1),\bar{\theta}} \end{cases},$$

where $C_{x_g,\underline{\theta}}$ and $C_{x_g,\bar{\theta}}$ are the limits of the basic function $A_{x_g}(\theta)$ when θ tends to negative and positive infinities respectively. If we have

$$\begin{cases} \lim_{\lambda_{x_g}\to-\infty} M_1(\theta-\lambda_{x_g}) = 1 \\ \lim_{\lambda_{x_g}\to\infty} M_1(\theta-\lambda_{x_g}) = 0 \end{cases}, \tag{5-1-8}$$

the first and second formulas of (5-1-7) include $C_{m_g,\underline{\theta}}$ and $C_{0,\bar{\theta}}$ respectively. The proof is easily obtained by expanding (5-1-5). In order that (5-1-8) should be true, it is necessary and sufficient that the lower and upper asymptotes of $M_1(\theta)$ should be zero and unity respectively. In any case, both $C_{0,\underline{\theta}}$ and $C_{m_g,\bar{\theta}}$ are zero.

For the purpose of illustration, suppose three formulas, the normal ogive function, the logistic function, and the square-logistic function, are given as $P^*_{x_g}(\theta)$, such that

$$P^*_{x_g}(\theta) = \Phi(b_{x_g}, 1/a_g^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a_g(\theta-b_{x_g})} e^{-(t^2/2)}\,dt, \tag{5-1-9}$$

$$P^*_{x_g}(\theta) = [1+e^{-Da_g(\theta-b_{x_g})}]^{-1}, \tag{5-1-10}$$

and

$$P^*_{x_g}(\theta) = [1+e^{-Da_g(\theta-b_{x_g})}]^{-2}, \tag{5-1-11}$$

where a_g (>0) is an item parameter, b_{x_g} is a parameter assigned to the score category x_g , and $D(>0)$ is a scaling factor. Then the asymptotic basic functions are given by

$$\tilde{A}_{x_g}(\theta) = -a_g^2(\theta-b_{x_g}), \tag{5-1-12}$$

$$\tilde{A}_{x_g}(\theta) = \frac{Da_g[1-e^{Da_g(\theta-b_{x_g})}]}{1+e^{Da_g(\theta-b_{x_g})}} \tag{5-1-13}$$

and

$$\tilde{A}_{x_g}(\theta) = \frac{Da_g[2-e^{Da_g(\theta-b_{x_g})}]}{1+e^{Da_g(\theta-b_{x_g})}} \tag{5-1-14}$$

respectively. Thus we can easily see that the asymptotic basic function derived from the normal ogive function is a straight line with positive and negative infinities as its asymptotic values, whereas the other two asymptotic functions are curves with finite values, Da_g and $-Da_g$, and $2Da_g$ and $-Da_g$, as their asymptotic values respectively. It also is easily seen from these formulas that the asymptotic basic functions derived from the normal ogive and logistic functions have $(b_{x_g}, 0)$ as their centers of symmetry, whereas the asymptotic basic function derived from the square-logistic function has

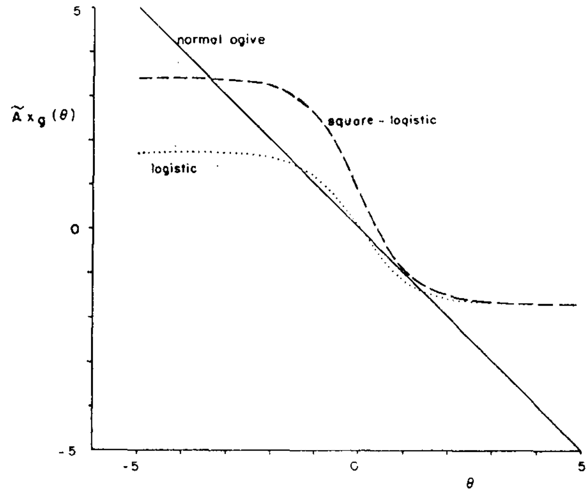

FIGURE 5-1-1

Three examples of the asymptotic basic function given by (5-1-12), (5-1-13) and (5-1-14) respectively. $P^*_{x_g}(\theta)$ for these examples are the normal ogive function, the logistic function, and the square logistic function, which are given by (5-1-9), (5-1-10) and (5-1-11) respectively, with parameter values, $a_g = 1$, $b_{x_g} = 0$, and $D = 1.7$.

$(b_{x_g}, Da_g/2)$ as its center of symmetry. Figure 5-1-1 presents these three asymptotic basic functions with the parameter values, $a_g = 1$ and $b_{x_g} = 0$, and the value of the scaling factor, $D = 1.7$. It is worthy to note that the two curves derived from the normal ogive and logistic functions are remarkably different, in spite of the fact that $P^*_{x_g}(\theta)$ for these models are so much alike when $D = 1.7$ [Birnbaum, 1968]. As another example, we shall consider the case in which $P^*_{x_g}(\theta)$ is given by

$$P^*_{x_g}(\theta) = 1 - \exp\{-c_g e^{a_g(\theta - b_{x_g})}\}, \tag{5-1-15}$$

where $a_g(>0)$ and $c_g(>0)$ are item parameters, and b_{x_g} is a parameter assigned to the score category x_g. In this example, the asymptotic basic function is given by

$$\tilde{A}_{x_g}(\theta) = a_g[1 - c_g e^{a_g(\theta - b_{x_g})}], \tag{5-1-16}$$

which provides a strictly decreasing and asymmetric curve, with a_g and negative infinity as its upper and lower asymptotes, taking $a_g(1 - c_g)$ when $\theta = b_{x_g}$.

5.2 Sufficient Condition for the Score Categories to be Syndrome Response Categories

In the homogeneous case of graded response level, it is obvious that Situation B (cf. Section 3.2) cannot occur, and Situation A is the sole possibility for all the $(m_g + 1)$ score categories to item g to be syndrome response categories, when m_g is finite. Thus in this situation $M_1(\theta)$ has zero and unity as its lower and upper asymptotes.

One characteristic of the homogeneous case is that all the $(m_g + 1)$ score categories to item g are syndrome response categories, if $M_1(\theta)$, and hence $P^*_{x_g}(\theta)$ for the categories, 1 through m_g, with these asymptotes satisfies (5-1-3) and the asymptotic basic function, $\tilde{A}_{x_g}(\theta)$, which was defined in the preceding section, is strictly decreasing in θ, or

$$\text{(5-2-1)} \qquad \frac{\partial}{\partial\theta} \tilde{A}_{x_g}(\theta) \leq 0,$$

where an equality holds at most at an enumerable number of points. The proof will be obtained by replacing θ for x and $P^*_{x_g}$ for f in Appendix. Thus if (5-2-1) is true, the basic function $A_{x_g}(\theta)$ is strictly decreasing in θ, *i.e.*, it satisfies (2-1). It is easily seen that, if (5-2-1) holds for any one category, 1 through m_g, it also holds for all the other $(m_g - 1)$ categories, because of the identity of their asymptotic basic functions except for their positions on the trait continuum. In fact, we can conceive of infinitely many combinations of m_g $P^*_{x_g}(\theta)$'s ,with a constraint (4-1-8).

Actually (2-1) implies (5-1-3), for, if $\partial/\partial\theta\ M_1(\theta) = 0$ at $\theta = \theta_0$, we also have $A_0(\theta) = 0$ at $\theta = \theta_0$, and, on the other hand, $C_{0,g} = 0$, as was mentioned in the preceding section. As a result, $A_0(\theta)$ cannot be strictly decreasing in θ. Similar discussion can also be made for category m_g. Thus (2-1) implies (5-1-3), and Situation A cannot be reached *unless the asymptotic basic function exists.*

It is worth noting that (5-2-1) implies that the first derivative of $M_1(\theta)$, and hence of $P^*_{x_g}(\theta)$ for the categories, 1 through m_g, should be *uni-modal*, and, moreover, its second derivative should not equal zero except at the modal point. For since $\partial/\partial\theta\ P^*_{x_g}(\theta)$ is positive and approaches zero as θ tends to negative and positive infinities respectively, it should have at least one modal point, and at that point $\partial^2/\partial\theta^2\ P^*_{x_g}(\theta)$ equals zero; and if $\partial^2/\partial\theta^2\ P^*_{x_g}(\theta)$ equals zero at more than one point of θ, $\tilde{A}_{x_g}(\theta)$ also equals zero at these points of θ, and, therefore, $\tilde{A}_{x_g}(\theta)$ cannot be strictly decreasing in θ.

Thus we have seen that the condition which makes all the score categories of a specified item syndrome response categories is much simplified in the homogeneous case. For the purpose of illustration, we shall consider the two examples, in which $P^*_{x_g}(\theta)$ is given by (5-1-9) and (5-1-10) in the preceding section, *i.e.*, the normal ogive and logistic functions. It has been demonstrated by Samejima [Samejima, 1969, Chapter 5] that in these two cases Situation A is provided, by directly proving the satisfaction of (2-1) in each case. It is also possible, however, to prove this through the satisfaction of (5-2-1) as follows. Since the asymptotic basic functions are given by (5-1-12) and (5-1-13) respectively, we obtain

(5-2-2)
$$\frac{\partial}{\partial \theta} \tilde{A}_{x_g}(\theta) = -a_g^2 < 0$$

and

(5-2-3)
$$\frac{\partial}{\partial \theta} \tilde{A}_{x_g}(\theta) = \frac{-2D^2 a_g^2 e^{Da_g(\theta - b_{x_g})}}{[1 + e^{Da_g(\theta - b_{x_g})}]^2} < 0,$$

which are satisfactions of (5-2-1). The proof is simpler in these two examples if we use (5-2-1), instead of (2-1).

We shall further proceed in these two examples to find out what kinds of functions for $M_{x_g}(\theta)$ these two models have. Since $M_1(\theta)$ equals $P_1^*(\theta)$, our interest is in the categories, 2 through m_g. In the normal ogive model, we can write from (4-5-1) and (5-1-9)

(5-2-4)
$$M_{x_g}(\theta) = \frac{\int_{-\infty}^{a_g(\theta - b_{x_g})} e^{-(t^2/2)}\, dt}{\int_{-\infty}^{a_g(\theta - b_{(x_g-1)})} e^{-(t^2/2)}\, dt} = 1 - \frac{\int_{a_g(\theta - b_{x_g})}^{a_g(\theta - b_{(x_g-1)})} e^{-(t^2/2)}\, dt}{\int_{-\infty}^{a_g(\theta - b_{(x_g-1)})} e^{-(t^2/2)}\, dt}.$$

Since the second term of the rightest hand side of (5-2-4) tends to zero as θ tends to positive infinity, the upper asymptote of $M_{x_g}(\theta)$ is unity, which is consistent with the necessary condition for Situation A. The first derivatives of the numerator and denominator of the second rightest hand side of (5-2-4) are given by

(5-2-5)
$$\begin{cases} a_g \exp\{-a_g^2(\theta - b_{x_g})^2/2\} \\ a_g \exp\{-a_g^2(\theta - b_{(x_g-1)})^2/2\} \end{cases}$$

respectively, so that by L'Hospital's Rule we obtain

(5-2-6)
$$\lim_{\theta \to -\infty} M_{x_g}(\theta) = \exp\{-a_g^2(b_{x_g}^2 - b_{(x_g-1)}^2)/2\} \cdot [\lim_{\theta \to -\infty} \exp\{a_g^2(b_{x_g} - b_{(x_g-1)})\theta\}] = 0.$$

Thus, together with the observation made in Section 4.5, it has been made clear that in the normal ogive model $M_{x_g}(\theta)$ is a strictly increasing function of θ with zero and unity as its lower and upper asymptotes for the categories, 2 through m_g, as well as for category 1. The upper graph of Figure 5-2-1 illustrates with $P_{x_g}^*(\theta)$ by solid line and $M_{x_g}(\theta)$ by dashed line with an exception of $M_1(\theta)$, for the case where $m_g = 6$, $a_g = 1$, and $b_{x_g} = -5.5, -5.0, -4.0, -2.0, 1.0, 5.0$ respectively in the normal ogive model. Note that $M_1(\theta) = P_1^*(\theta)$. We can see in this figure that for the two categories, 5 and 6, $M_{x_g}(\theta)$ practically coincides with $P_{x_g}^*(\theta)$, which indicates that the normal ogive model $M_{x_g}(\theta)$ practically equals $P_{x_g}^*(\theta)$ if the distance between $b_{(x_g-1)}$ and b_{x_g} is more than 3.0 by the standard deviation unit.

In the logistic model, we can write from (4-5-1) and (5-1-10) that

$$(5\text{-}2\text{-}7)\qquad \begin{aligned} M_{x_g}(\theta) &= [1 + e^{-Da_g(\theta - b_{x_g})}]^{-1}[1 + e^{-Da_g(\theta - b_{(x_g-1)})}] \\ &= 1 - [1 - e^{-Da_g(b_{x_g} - b_{(x_g-1)})}][1 + e^{Da_g(\theta - b_{x_g})}]^{-1}. \end{aligned}$$

Thus the lower and upper asymptotes are given by

$$(5\text{-}2\text{-}8)\qquad \begin{cases} \lim\limits_{\theta \to -\infty} M_{x_g}(\theta) = \exp\{-Da_g(b_{x_g} - b_{(x_g-1)})\} \\ \lim\limits_{\theta \to \infty} M_{x_g}(\theta) = 1 \end{cases},$$

which indicates that in the logistic model the lower asymptote of $M_{x_g}(\theta)$ is greater than zero, except for category 1. This value is a strictly decreasing

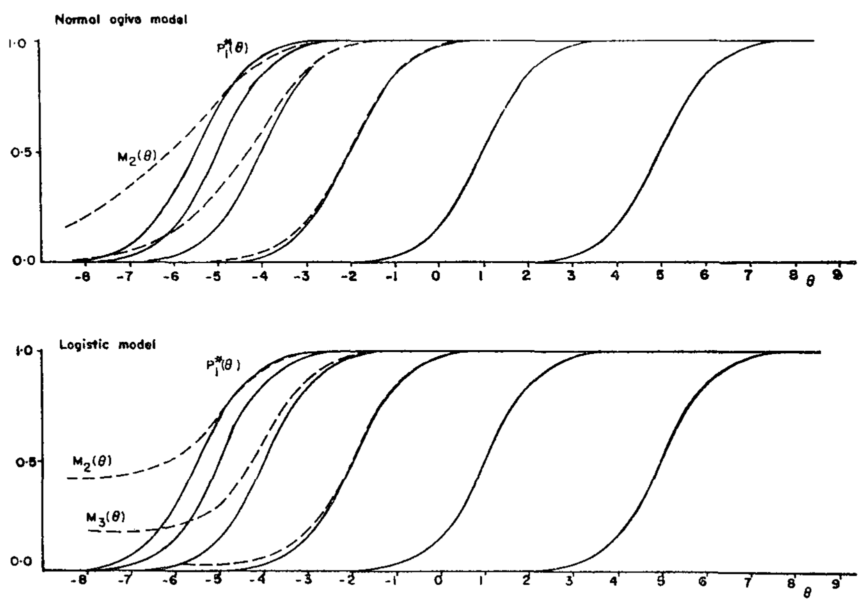

FIGURE 5–2–1

The normal ogive model (the upper graph) and the logistic model (the lower graph) of the graded item response, where $m_g = 6$, and $a_g = 1$, $b_{x_g} = -5.5, -5.0, -4.0, -2.0, 1.0, 5.0$, and $D = 1.7$ respectively. In both graphs, $P_{x_g}^*(\theta)$ for the categories, 1 through 6, (solid line), and $M_{x_g}(\theta)$ for the categories, 2 through 6, (dashed line) are presented.

function of the distance, $[b_{x_g} - b_{(x_g-1)}]$, which tends to unity as the distance tends to zero, and tends to zero as it tends to positive infinity.

We have already seen in Section 4.2 that, in order to obtain Situation A, the lower asymptote of $M_{x_g}(\theta)$ does not have to be zero, except for category 1. Thus the logistic model in the homogeneous case is a typical example of such a situation. The lower graph of Figure 5-2-1 presents $P^*_{x_g}(\theta)$ and $M_{x_g}(\theta)$ in the logistic model by solid and dashed lines respectively. In this example, the parameter values are the same as those used in the previous example of the normal ogive model, and $D = 1.7$, which makes $P^*_{x_g}(\theta)$ very close to the one in the normal ogive model with the same parameter values [Birnbaum, 1968, page 399]. The values of the lower asymptote of $M_{x_g}(\theta)$ are approximately 0.427, 0.182, 0.033, 0.006 and 0.001 for the categories, 2 through 6, respectively.

It is interesting to note that, in spite of the close similarity of $P^*_{x_g}(\theta)$ between the two examples, $M_{x_g}(\theta)$ is remarkably different in the two models. Similarities can be found only for categories 5 and 6, for which $M_{x_g}(\theta)$ is practically equal to $P^*_{x_g}(\theta)$ in the logistic model also. In both models $M_{x_g}(\theta)$ tends to unity for all θ as the distance between b_{x_g} and $b_{(x_g-1)}$ tends to zero.

A possible explanation of $M_{x_g}(\theta)$ in the logistic model may be that the closer the psychological distance between two score categories the more likely for the subject to be attracted by the higher of the two, if he has already been attracted by the lower one. Since the model is of interest, we shall further inquire into the components of the basic function in this model. Since the lower asymptote of $M_{x_g}(\theta)$ is greater than zero for the categories, 2 through m_g, the first derivative of log $M_{x_g}(\theta)$ cannot be strictly decreasing in θ, though altogether they satisfy the first formula of (4-2-1) for syndrome response categories. Then what kind of function does $M_{x_g}(\theta)$ provide for these categories? Since, in general, we have

$$\text{(5-2-9)} \qquad \frac{\partial}{\partial\theta} \log M_{x_g}(\theta) = \frac{\partial}{\partial\theta} \log P^*_{x_g}(\theta) - \frac{\partial}{\partial\theta} \log P^*_{(x_g-1)}(\theta),$$

we can write from (5-1-10) for the logistic model

$$\text{(5-2-10)} \qquad \frac{\partial}{\partial\theta} \log M_{x_g}(\theta) = Da_g[P^*_{(x_g-1)}(\theta) - P^*_{x_g}(\theta)].$$

Thus the first derivative of log $M_{x_g}(\theta)$ turns out to be a unimodal and symmetric function of θ, with the modal point, $\theta_{\max}$, such that

$$\text{(5-2-11)} \qquad \theta_{\max} = \tfrac{1}{2}[b_{x_g} + b_{(x_g-1)}],$$

and its maximum value is given by

$$Da_g[1 - \sqrt{G_{x_g}}][1 + \sqrt{G_{x_g}}]^{-1},$$

where G_{x_g} is the lower asymptote of $M_{x_g}(\theta)$ given by (5-2-8). From the definition of G_{x_g} it is easily seen that this maximum value is a strictly increasing function of the distance, $[b_{x_g} - b_{(x_g-1)}]$, with zero and Da_g as its two limits. It is also obvious from (5-2-10) that the two lower asymptotes of the first derivative of log $M_{x_g}(\theta)$ are zero. On the other hand, since we have from (5-2-7)

$$(5\text{-}2\text{-}12) \qquad \frac{\partial}{\partial\theta} M_{x_g}(\theta) = Da_g[1 - G_{x_g}]P^*_{x_g}(\theta)[1 - P^*_{x_g}(\theta)],$$

we can write from this and (5-2-7)

$$(5\text{-}2\text{-}13) \qquad \frac{\partial}{\partial\theta} \log [1 - M_{(x_g+1)}(\theta)] = -Da_g P^*_{(x_g+1)}(\theta)$$

$$= \frac{\partial}{\partial\theta} \log [1 - P^*_{(x_g+1)}(\theta)]$$

for the second formula of (4-2-1). Thus it is obvious that this component of the basic function is strictly decreasing in θ with zero and $-Da_g$ as its upper and lower asymptotes. As for the first derivative of log $M_1(\theta)$, since $M_1(\theta)$ equals $P^*_1(\theta)$, it is a strictly decreasing function of θ with Da_g and zero as its upper and lower asymptotes [Samejima, 1969, pages 34–35]. Figure 5-2-2 illustrates with a basic function and its components in the logistic model, where m_g is greater than 4, $x_g = 4$, $D = 1.7$. $a_g = 1.0$, and $b_{x_g} = -2.5$, -2.0, 0.0, 1.0, 3.0 for the categories, 1 through 5, respectively. The curve drawn by dashed line is the basic function, *i.e.*, the sum total of its five components, each of which is drawn by solid line.

We shall observe the two other examples given in the preceding section. For these examples, $P^*_{x_g}(\theta)$ are given by (5-1-11) and (5-1-15) respectively. From (5-1-14) and (5-1-16) of the asymptotic basic functions of these examples, we obtain

$$(5\text{-}2\text{-}14) \qquad \frac{\partial}{\partial\theta} \tilde{A}_{x_g}(\theta) = \frac{-3D^2a_g^2e^{Da_g(\theta-b_{x_g})}}{[1 + e^{Da_g(\theta-b_{x_g})}]^2} < 0$$

and

$$(5\text{-}2\text{-}15) \qquad \frac{\partial}{\partial\theta} \tilde{A}_{x_g}(\theta) = -a_g^2c_ge^{a_g(\theta-b_{x_g})} < 0$$

respectively, which satisfy (5-2-1). Since $P^*_{x_g}(\theta)$ is strictly increasing in θ with zero and unity as their lower and upper asymptotes in these two models, we can conclude that these two models also provide Situation A. This conclusion has already been reached in the former example through the direct satisfaction of (2-1) [Samejima, 1969, page 88].

It is easily seen that, if all the $(m_g + 1)$ score categories of item g are syndrome response categories in the homogeneous case, $M_{x_g}(\theta)$ is strictly

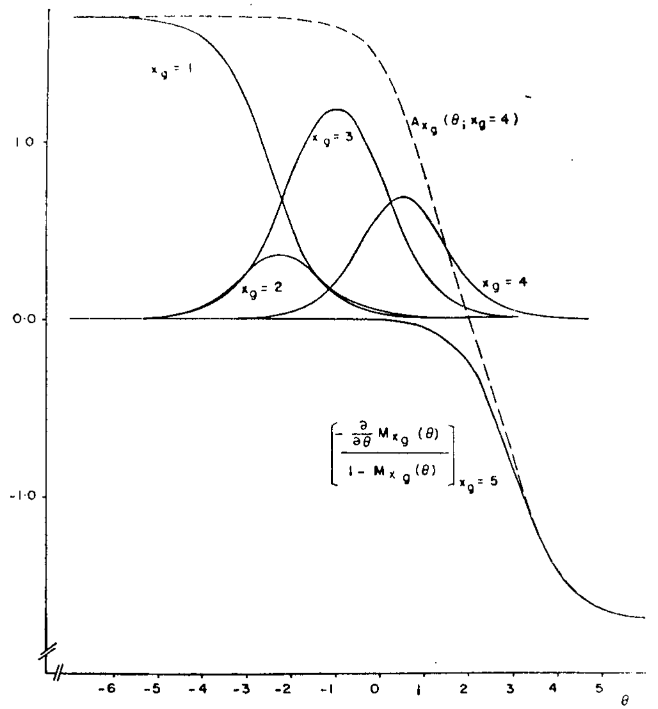

FIGURE 5-2-2

A basic function (dashed line) and its five components (solid line) of the logistic model of the graded item response, where $D = 1.7$, $a_g = 1.0$, and $b_{x_g} = -2.5, -2.0, 0.0, 1.0, 3.0$ for the categories, 1 through 5. The first four components are the derivatives of $\log M_1(\theta)$, through $\log M_4(\theta)$, and the last component is the derivative of $\log \{1 - M_5(\theta)\}$, and the basic function is $A_4(\theta)$.

increasing in θ not only for category 1, but also for the categories, 2 through m_g. To prove this, we have from (4-1-7)

$$\frac{\partial}{\partial\theta} \log P^*_{x_g}(\theta) = \frac{\partial}{\partial\theta} \log M_1(\theta - \lambda_{x_g}), \tag{5-2-16}$$

where λ_{x_g} satisfies (4-1-8), for the categories, 1 through m_g. Since (5-2-16) equals the basic function when $x_g = m_g$, it is strictly decreasing in θ for all these categories, 1 through m_g. From this fact and (5-2-16) we obtain

$$\frac{\partial}{\partial\theta} \log P^*_{(x_g-1)}(\theta) < \frac{\partial}{\partial\theta} \log P^*_{x_g}(\theta) \tag{5-2-17}$$

for the categories, 2 through m_g, which satisfies (4-5-3), *i.e.*, the necessary and sufficient condition for $M_{x_g}(\theta)$ to be strictly increasing in θ, which was discussed in Section 4.5. Thus it has been proved that $M_{x_g}(\theta)$ is strictly increasing in θ for the categories, 1 through m_g, provided that all the $(m_g + 1)$ score categories of item g are syndrome response categories.

5.3 Orderliness and Reclassification of Syndrome Score Categories

As was observed in Section 4.3, in the heterogeneous case the modal points of syndrome score categories are not always arranged in the order of category scores, even in Situation A. As distinct from this, in the homogeneous case, since the asymptotic basic function always exists in Situation A, if the asymptotic basic function satisfies (5-2-1), the above statement is always true, *i.e.*, starting from the terminal maximum at negative infinity for category 0, $(m_g - 1)$ local maxima follow in the order of their category scores, and finally comes the terminal maximum at positive infinity for category m_g.

The proof can be given from a more general standpoint. Suppose that $P^*_{x_g}(\theta)$ is fixed for an arbitrary x_g, which is greater than 1 and less than m_g. We can write from (1-2-5), (4-1-4) and (4-1-7)

(5-3-1)
$$A_{x_g}(\theta) = \frac{\int_{\theta-\beta}^{\theta} \frac{\partial^2}{\partial t^2} P^*_{x_g}(t)\, dt}{\int_{\theta-\beta}^{\theta} \frac{\partial}{\partial t} P^*_{x_g}(t)\, dt},$$

where

(5-3-2)
$$\beta = \lambda_{(x_g+1)} - \lambda_{x_g}.$$

As was observed in the preceding section, if (5-2-1) is true, we can conceive of infinitely many possible $P^*_{(x_g+1)}(\theta)$ by adjusting the value of $\lambda_{(x_g+1)}$, which uniformly provide a syndrome response category for x_g. Let β_1 and β_2 be arbitrary positive values satisfying

(5-3-3)
$$\beta_1 > \beta_2.$$

Hereafter, we shall denote the basic functions of category x_g obtainable by using β_1 and β_2 in (5-3-1) by $A_{x_g}(\theta; \beta_1)$ and $A_{x_g}(\theta; \beta_2)$ respectively. By following the logic used in Appendix, it is easily seen that

(5-3-4)
$$A_{x_g}(\theta; \beta_1) > A_{x_g}(\theta; \beta_2)$$

throughout the whole range of θ. On the other hand, if we consider category $(x_g - 1)$, we can write

(5-3-5)
$$A_{(x_g-1)}(\theta) = \frac{\int_{\theta}^{\theta+\gamma} \frac{\partial^2}{\partial t^2} P^*_{x_g}(t)\, dt}{\int_{\theta}^{\theta+\gamma} \frac{\partial}{\partial t} P^*_{x_g}(t)\, dt},$$

where

$$\gamma = \lambda_{x_g} - \lambda_{(x_g-1)} \, . \tag{5-3-6}$$

In a similar manner, we shall define $A_{(x_g-1)}(\theta;\gamma_1)$ and $A_{(x_g-1)}(\theta;\gamma_2)$ for an arbitrary pair of positive numbers satisfying

$$\gamma_1 > \gamma_2 \, . \tag{5-3-7}$$

Then, by following the logic used in Appendix again, it is easily seen that

$$A_{(x_g-1)}(\theta;\gamma_1) < A_{(x_g-1)}(\theta;\gamma_2) \tag{5-3-8}$$

for all θ. Since we have

$$\lim_{\beta\to 0} A_{x_g}(\theta) = \lim_{\gamma\to 0} A_{(x_g-1)}(\theta) = \tilde{A}_{x_g}(\theta), \tag{5-3-9}$$

we can write from this and (5-3-4) and (5-3-8)

$$A_{(x_g-1)}(\theta;\gamma_1) < A_{(x_g-1)}(\theta;\gamma_2) \le \tilde{A}_{x_g}(\theta) \le A_{x_g}(\theta;\beta_2) < A_{x_g}(\theta;\beta_1), \tag{5-3-10}$$

and from this, we obtain, in general

$$A_{(x_g-1)}(\theta) < \tilde{A}_{x_g}(\theta) < A_{x_g}(\theta). \tag{5-3-11}$$

In virtue of the fact that the modal point of $P_{x_g}(\theta)$ is the value of θ at which $A_{x_g}(\theta) = 0$, generalizing this to include cases where β and γ are positive infinity, the orderliness of the modal points of $P_{x_g}(\theta)$ has been proved completely.

The formulas (5-3-4) and (5-3-8) also have other implications. Suppose there is a syndrome response pattern of n items. If we pick up one of these items and change its scoring strategy, then what will be the effect on its maximum likelihood estimate? Suppose that we pick up item g whose score category in the response pattern is x_g. If we fix $P^*_{x_g}(\theta)$ and change the value of β to a greater one so that $P^*_{(x_g+1)}(\theta)$ is shifted to the positive direction on the trait continuum, the maximum *likelihood* estimate of the new response pattern is greater than the original one. On the other hand, if we fix $P^*_{(x_g+1)}(\theta)$ and shift $P^*_{x_g}(\theta)$ to the negative direction on the trait continuum so that a new syndrome response category is provided for x_g, the maximum likelihood estimate of the resulting response pattern is less than the original one.

Figure 5-3-1 illustrates with three examples of $A_{x_g}(\theta)$ and $A_{(x_g-1)}(\theta)$ each when $P^*_{x_g}(\theta)$ is the standard normal ogive function, where β's and γ's are uniformly 2, 4 and 6, by solid line, whereas the asymptotic basic function, $\tilde{A}_{x_g}(\theta)$, is drawn by dotted line in the same figure. We can see that the effects of β are conspicuous for the positive values of θ in this example, while those of γ are conspicuous for the negative values of θ.

As for the reclassification of syndrome score categories, it is obvious from

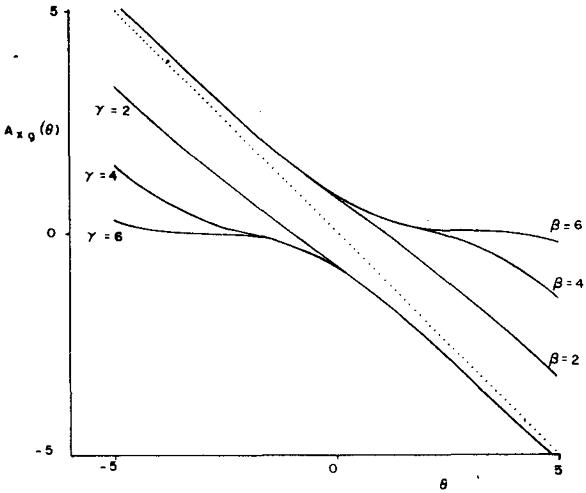

FIGURE 5–3–1

Three examples of $A_{x_g}(\theta)$ and $A_{(x_g-1)}(\theta)$ each (solid line) when $P^*_{x_g}(\theta)$ is the standard normal orgive function, where β in (5-3-1) are 2, 4 and 6, and γ in (5-3-5) are also 2, 4 and 6 respectively. Also the asymtotic basic function (dotted line) is given.

the observation made in the preceding section that, if the asymptotic basic function satisfies (5-2-1), any combination of adjacent syndrome score categories provides a new syndrome score category. Thus in the homogeneous case, if the asymptotic basic function is strictly decreasing in θ and $P^*_{x_g}(\theta)$ has zero and unity as its lower and upper asymptotes, we can freely rescore the data by combining two or more adjacent categories, while keeping Situation A. In such a case, it is obvious from the earlier observation that the maximum likelihood estimate will be shifted to the negative direction if the original score category is combined with lower adjacent categories, and it will be shifted to the positive direction if the original one is combined with higher adjacent categories.

5.4 Positive-Exponent Family

So far we have observed various characteristics of the homogeneous case, especially when all the score categories of item g are syndrome response categories, and consequently Situation A occurs. We shall see in this section that, given a specified function for $P^*_{x_g}(\theta)$ which provides Situation A, we can generate a family of $P^*_{x_g}(\theta)$ all of whose members provide Situation A.

Suppose $F(\theta)$ is a three-times differentiable function of θ, which satisfies

(5-4-1) $$\frac{\partial}{\partial \theta} F(\theta) = f(\theta) > 0$$

and

$$(5\text{-}4\text{-}2) \qquad \begin{cases} \lim_{\theta \to -\infty} F(\theta) = 0 \\ \lim_{\theta \to \infty} F(\theta) = 1 \end{cases}.$$

It is easily seen that this assumption implies

$$(5\text{-}4\text{-}3) \qquad F(\theta) = \int_{-\infty}^{\theta} f(t)\, dt$$

and

$$(5\text{-}4\text{-}4) \qquad \lim_{\theta \to -\infty} f(\theta) = \lim_{\theta \to \infty} f(\theta) = 0.$$

Suppose, further, that

$$(5\text{-}4\text{-}5) \qquad \frac{\partial^2}{\partial \theta^2} \log f(\theta) \leq 0,$$

where an equality holds at most at an enumerable number of points. From this and (5-4-1) and (5-4-4) we obtain

$$(5\text{-}4\text{-}6) \qquad \begin{cases} \lim_{\theta \to -\infty} \dfrac{\partial}{\partial \theta} \log f(\theta) > 0 \\ \lim_{\theta \to \infty} \dfrac{\partial}{\partial \theta} \log f(\theta) < 0 \end{cases}.$$

For convenience, using the same symbol as in (4-1-4), we shall define $P^*_{x_g}(\theta)$ such that

$$(5\text{-}4\text{-}7) \qquad \begin{aligned} P^*_{x_g}(\theta) &= [F\{a_g(\theta - b_{x_g})\}]^s \\ &= s \int_{-\infty}^{a_g(\theta - b_{x_g})} [F(t)]^{s-1} f(t)\, dt \end{aligned}$$

for

$$(5\text{-}4\text{-}8) \qquad s \geq 1,$$

where a_g is a positive constant and b_{x_g} is a real constant, and call this family of $P^*_{x_g}(\theta)$ a *positive-exponent family*. Thus given a $F(\theta)$ which satisfies (5-4-1), (5-4-2) and (5-4-5), we can generate a positive-exponent family, which contains a non-enumerable number of members.

It is easily seen from (5-4-2) and (5-4-7) that

$$(5\text{-}4\text{-}9) \qquad \begin{cases} \lim_{\theta \to -\infty} P^*_{x_g}(\theta) = 0 \\ \lim_{\theta \to \infty} P^*_{x_g}(\theta) = 1 \end{cases}$$

and

$$\text{(5-4-10)}\qquad \frac{\partial}{\partial \theta} P^*_{x_g}(\theta) = s a_g [F(\theta^*)]^{s-1} f(\theta^*) > 0,$$

where

$$\text{(5-4-11)}\qquad \theta^* = a_g(\theta - b_{x_g}),$$

which indicate that $P^*_{x_g}(\theta)$ defined by (5-4-7) is strictly increasing in θ with zero and unity as its lower and upper asymptotes. Since we have from (5-4-10)

$$\text{(5-4-12)}\qquad \frac{\partial^2}{\partial \theta^2} P^*_{x_g}(\theta) = a_g \left[(s-1) \frac{\partial}{\partial \theta^*} \log F(\theta^*) + \frac{\partial}{\partial \theta^*} \log f(\theta^*) \right] \frac{\partial}{\partial \theta} P^*_{x_g}(\theta),$$

for the asymptotic basic function we obtain

$$\text{(5-4-13)}\qquad \tilde{A}_{x_g}(\theta) = a_g \left[(s-1) \frac{\partial}{\partial \theta^*} \log F(\theta^*) + \frac{\partial}{\partial \theta^*} \log f(\theta^*) \right].$$

It is obvious from this and (5-4-5) and (5-4-8), and the observation made in Section 5.2, that this asymptotic basic function satisfies (5-2-1), *i.e.*, it is strictly decreasing in θ. As was observed in Section 5.2, this means that any member of the positive-exponent family defined by (5-4-7) provides a set of $(m_g + 1)$ syndrome score categories, for an arbitrary positive integer m_g and arbitrary values of distances between λ_{x_g} and $\lambda_{(x_g+1)}$. The operating characteristic of a score category thus obtained is given by

$$\text{(5-4-14)}\qquad P_{x_g}(\theta) = s \int_{a_g(\theta - b_{(x_g+1)})}^{a_g(\theta - b_{x_g})} [F(t)]^{s-1} f(t)\, dt,$$

and the basic function is, as defined, the ratio of its first derivative to $P_{x_g}(\theta)$ itself. The asymptotic values of the basic function, $C_{x_g,\underline{\theta}}$ and $C_{x_g,\bar{\theta}}$, are obtained from (5-1-7), (5-4-6) and (5-4-13) such that

$$\text{(5-4-15)}\qquad \begin{cases} C_{x_g,\underline{\theta}} = s a_g \left[\lim_{\theta \to -\infty} \frac{\partial}{\partial \theta} \log f(\theta) \right] > 0 \\ C_{x_g,\bar{\theta}} = a_g \left[\lim_{\theta \to \infty} \frac{\partial}{\partial \theta} \log f(\theta) \right] < 0 \end{cases},$$

with exceptions

$$\text{(5-4-16)}\qquad \begin{cases} C_{0,\underline{\theta}} = 0 \\ C_{m_g,\bar{\theta}} = 0 \end{cases}.$$

It is worth noting that, if $[F(\theta) - 0.5]$ is an odd function, *i.e.*, if $F(\theta)$ is symmetric with (0, 0.5) as its center of symmetry, all the members of its positive-exponent family are *asymmetric* except for the case where $s = 1$.

For in such a case we can write from (5-4-7)

$$P^*_{x_g}(\theta) = [1 - F\{a_g(-\theta + b_{x_g})\}]^s \neq 1 - [F\{a_g(-\theta + b_{x_g})\}]^s = 1 - P^*_{x_g}(-\theta + 2b_{x_g}), \tag{5-4-17}$$

unless $s = 1$. The square logistic function, which was used as an example of asymmetric curves by Samejima [Samejima, 1969, pages 87–91] and also used as an example in previous sections, is a typical example of members of a positive-exponent family, when $F(\theta)$ is a logistic function. As was pointed out by Samejima [Samejima, 1969, Chapter 10], there exists a paradox when we use a symmetric curve for the item characteristic function $P_g(\theta)$ of the dichotomous case. This comes from the fact that, if we use a symmetric function as $P_g(\theta)$, the likelihood functions with respect to a pair of symmetric response patterns are also symmetric. For illustrative purposes, we shall consider a simple case where three items are scored dichotomously, *i.e.*, success or failure, and the item characteristic function of each item is a normal ogive function given by (5-1-9) in which b_{x_g} is replaced by b_g, and the parameter values are

$$\begin{cases} a_1 = a_2 = a_3 = 1 \\ b_1 = -1 \\ b_2 = 0 \\ b_3 = 1 \end{cases} . \tag{5-4-18}$$

The six possible response patterns containing one or two successes will be denoted by R_{100}, R_{010}, R_{001}, R_{110}, R_{101} and R_{011}, where subscripts indicate item scores arranged in the order of item numbers. Figure 5-4-1 presents the likelihood functions for these six response patterns, which are the operating characteristics of the response patterns themselves. We can easily see that the likelihood functions are symmetric with each other for the three pa rs: R_{100}, R_{110}; R_{010}, R_{101}; and R_{001}, R_{011}; respectively. This does not occur if we use an asymmetric function for the item characteristic function of each item, and one of the utilities of the positive-exponent family is that it provides asymmetric formulas.

If we have

$$\frac{\partial^2}{\partial\theta^2} \log f(\theta) \leq \frac{\partial^2}{\partial\theta^2} \log F(\theta) \tag{5-4-19}$$

for a specified $F(\theta)$, where an equality holds at most at an enumerable number of points, its positive-exponent family can be expanded for the case where

$$0 < s < 1. \tag{5-4-20}$$

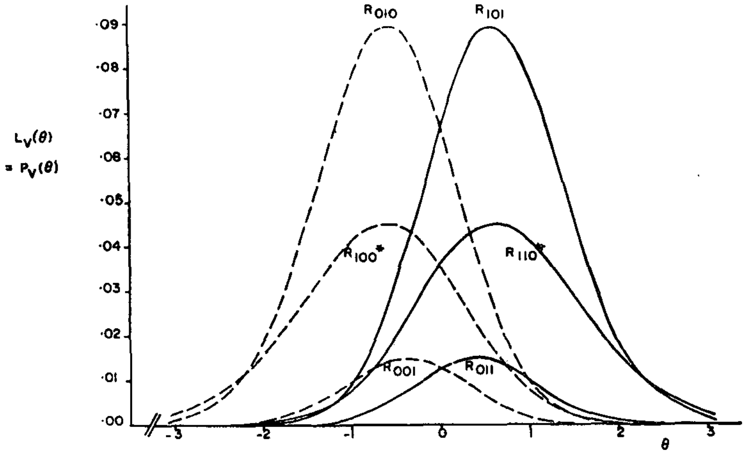

FIGURE 5-4-1

The likelihood functions for the six response patterns, R_{100}, R_{010}, R_{001}, R_{110}, R_{101} and R_{011}, to the hypothetical three items, based on the normal ogive model of the dichotomous response. Three curves (dashed line) are symmetric with the other three (solid line). The two curves with * represent the original values multiplied by 10^{-1}.

For, if (5-4-19) is true, the asymptotic basic function given by (5-4-13) is strictly decreasing in θ for any s satisfying (5-4-20), since both sides of (5-4-19) are negative, or equal to zero at most at an enumerable number of points. In other words, in such a case $P^*_{x_g}(\theta)$ defined by (5-4-7) provides syndrome score categories for *any* positive value of s.

For the purpose of illustration, we shall prove that, if the normal ogive function or the logistic function is used as $F(\theta)$, its positive-exponent family can be defined for any positive value of s. Suppose that $F(\theta)$ is given by the standard normal ogive function such that

(5-4-21) $$F(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-(t^2/2)}\, dt.$$

Then we have

(5-4-22) $$f(\theta) = \frac{1}{\sqrt{2\pi}} e^{-(\theta^2/2)}$$

and

(5-4-23) $$\frac{\partial}{\partial \theta} f(\theta) = -\theta f(\theta),$$

and from these two formulas we obtain

(5-4-24) $$\frac{\partial^2}{\partial \theta^2} \log f(\theta) = -1.$$

On the other hand, we can write from (5-4-23)

$$\frac{\partial^2}{\partial\theta^2}\log F(\theta) = -\frac{f(\theta)}{F(\theta)}\left[\theta + \frac{f(\theta)}{F(\theta)}\right]. \tag{5-4-25}$$

Now let us consider the truncated standard normal distribution by cutting off and ignoring the part of its probability density function lying to the right of θ, and let $\mu(\theta)$ and $\sigma^2(\theta)$ denote its mean and variance. We can write

$$\begin{aligned}\mu(\theta) &= \frac{1}{F(\theta)}\int_{-\infty}^{\theta} tf(t)\,dt \\ &= -\frac{f(\theta)}{F(\theta)},\end{aligned} \tag{5-4-26}$$

and

$$\begin{aligned}\sigma^2(\theta) &= \frac{1}{F(\theta)}\int_{-\infty}^{\theta} t^2 f(t)\,dt - [\mu(\theta)]^2 \\ &= \frac{1}{F(\theta)}\int_{-\infty}^{\theta} (t^2 - 1)f(t)\,dt + 1 - [\mu(\theta)]^2 \\ &= 1 - \frac{f(\theta)}{F(\theta)}\left[\theta + \frac{f(\theta)}{F(\theta)}\right].\end{aligned} \tag{5-4-27}$$

Since the variance of the truncated standard normal distribution is positive for all θ, the rightest hand side of (5-4-27) is also positive, and from this and (5-4-24) and (5-4-25) we obtain (5-4-19) with a strict inequality. Thus the positive-exponent family can be defined for any positive s, if $F(\theta)$ is given by (5-4-21), *i.e.*, the standard normal ogive function.

Figure 5-4-2 presents twelve members of the positive-exponent family defined for (5-4-21). In these examples

$$s = \tfrac{1}{16}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, 1, 2, 3, 4, 5, 6, 10, 20 \quad \text{respectively.}$$

In the case where $F(\theta)$ is given by a logistic function such that

$$F(\theta) = (1 + e^{-D\theta})^{-1}, \tag{5-4-28}$$

where $D > 0$, we have

$$f(\theta) = DF(\theta)\,[1 - F(\theta)] \tag{5-4-29}$$

and

$$\frac{\partial}{\partial\theta} f(\theta) = D[1 - 2F(\theta)]f(\theta). \tag{5-4-30}$$

From these formulas we obtain

$$\frac{\partial^2}{\partial\theta^2}\log F(\theta) = -Df(\theta) \tag{5-4-31}$$

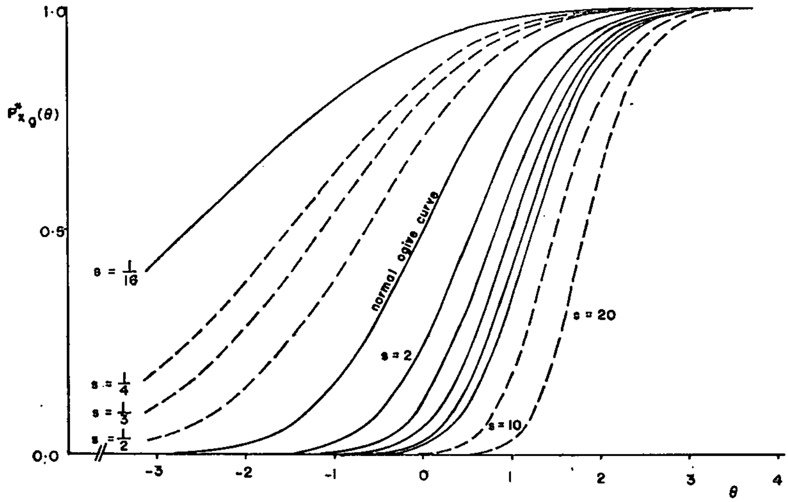

FIGURE 5-4-2

Twelve members of the positive-exponent family, where $F(\theta)$ is given by (5-4-21), *i.e.*, the standard normal ogive function, and $s = \frac{1}{16}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 10, 20$ respectively.

and

$$\text{(5-4-32)} \qquad \frac{\partial^2}{\partial\theta^2} \log f(\theta) = -2Df(\theta),$$

which satisfy (5-4-19) with a strict inequality for all θ. Thus the positive-exponent family can be defined for any positive s, if $F(\theta)$ is given by (5-4-28), *i.e.*, the logistic function.

The discussion of the positive-exponent family is also applicable for $W(\theta)$ introduced in Section 3.2. Thus we can generate $R_{k_g}(\theta)$ and $U_{k_g}(\theta)$ on the nominal response level from any member of the positive-exponent family of $W(\theta)$, which will be obtained by substituting W for F in (5-4-7), as well as $M_{x_g}(\theta)$ in the heterogeneous case of the graded response level, so that these functions should provide syndrome response categories in their own respective situations.

CHAPTER 6

THE DICHOTOMOUS RESPONSE LEVEL

The dichotomous response level is a special case of the graded level situation, in which $m_g = 1$. Thus $M_1(\theta)$, and hence $P_1^*(\theta)$ by virtue of (4-1-2) and (4-1-5), is the only function we should consider on this response level. Let $P_g(\theta)$ denote this function, which is called the item characteristic function [Lord & Novick, 1968] in the mental test theory, and the trace line [Lazarsfeld, 1959] in the latent structure analysis.

We can rewrite (2-1), which gives the definition of the syndrome score category, for categories 1 and 0 respectively, such that

$$\text{(6-1)} \qquad \begin{cases} \dfrac{\partial^2}{\partial \theta^2} \log P_g(\theta) \leq 0 \\ \\ \dfrac{\partial^2}{\partial \theta^2} \log\,[1 - P_g(\theta)] \leq 0 \end{cases},$$

where an equality holds at most at an enumerable number of points of θ in each formula. Note that this is the direct translation of (4-2-7), which was discussed in Section 4.2 of the heterogeneous case of the graded response level. Thus the lower and upper asymptotes of $P_g(\theta)$ should be zero and unity respectively, in order for $P_g(\theta)$ to provide syndrome score categories for categories, 0 and 1.

It is worth noting that the item characteristic function of the multiple-choice item, which is widely used in mental measurement [Birnbaum, 1968, pages 404-405] and takes the form

$$\text{(6-2)} \qquad P_g^*(\theta) = c_g + (1 - c_g)P_g(\theta),$$

where $P_g^*(\theta)$ is the item characteristic function of the multiple-choice item and c_g is a positive constant less than unity, *can never satisfy* the first formula of (6-1), since its lower asymptote is greater than 0. It provides a syndrome score category for category 0, but not for category 1, provided that $P_g(\theta)$ satisfies (6-1). Detailed discussion concerning this subject is made elsewhere [Samejima, in preparation [a]].

Since the dichotomous response level can be considered as a special case of the homogeneous case of the graded level situation, some of the important results in the homogeneous case are also valid in the dichotomous response level. If the asymptotic basic function, which is defined for $P_g(\theta)$ with zero and unity as its lower and upper asymptotes, satisfies (5-2-1), syndrome response categories are provided for both categories 0 and 1. All the members of a positive-exponent family can also be $P_g(\theta)$ which produce syndrome response categories on the dichotomous level.

One of the Rasch models of the item characteristic function [Rasch, 1960] is given by

$$P_g^*(\tau) = \left[1 + \frac{b_g^*}{\tau}\right]^{-1} \tag{6-3}$$

where τ is the trait in this case, having the domain

$$0 < \tau < \infty. \tag{6-4}$$

Transforming the variable τ into θ by

$$\theta = \log \tau \tag{6-5}$$

it is easily seen that (6-3) is the transformation of a special case of the logistic model, in which the discrimination index is assumed to be constant for all items [Birnbaum, 1968, page 402]. Thus by virtue of the transformation-free character of the maximum likelihood estimator this Rasch model also provides syndrome response categories for both categories 0 and 1, since the logistic model does.

CHAPTER 7

DISCUSSION

Throughout this paper a general model for free-response data was presented and discussed. First, it was emphasized that free-response data should be identified in terms of their contents rather than in terms of their formats. Then basic concepts and assumptions were given and *syndrome response patterns* and *categories* were defined. Three different levels of response categories, *i.e.*, the *nominal response level*, the *graded response level*, and the *dichotomous response level*, were specified and discussed separately. The graded response level was categorized into two cases, *heterogeneous* and *homogeneous*, and characteristics of each were observed. Conditions with which a given response category, or a set of $(m_g + 1)$ score categories, should be a syndrome response category or categories, were investigated on different response levels. In so doing two tendencies, being attracted by and the rejection of a given item response category, were taken into consideration, and fundamental formulas were established on them.

The graded response level is distinguished from the nominal response level in the sense that the distribution functions of the item score can be *stochastically ordered*. To be more precise, such a distribution function is given by

$$1 - P^*_{(x_g+1)}(\theta)$$

for a fixed value of θ, and this is strictly decreasing in θ for a fixed value of x_g, provided that $P^*_{(x_g+1)}(\theta)$ is strictly increasing in θ. This is true for both the homogeneous and heterogeneous cases. The modal points of the operating characteristics of syndrome score categories revealed themselves, however, possibly to be conspicuously disordered in the heterogeneous case, as distinct from those in the homogeneous case. This may be a characteristic difference between the two cases of the graded response level.

The present model can be considered as an expansion of the latent trait model, and terminologies were used in line with the latent trait theory. As the title shows, however, this model is fairly general, and the reader must not confine his imagination to a limited area of hypothetical constructs. We may, for instance, adopt our model in the study of memory using a one-to-one mapping of *time* as θ, the discussion of which will be made elsewhere.

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APPENDIX

Suppose $f(x)$ is three-times differentiable for the range

$$-\infty < x < \infty, \tag{A-1}$$

and strictly increasing, and its first derivative satisfies

$$\frac{d}{dx} f(x) > 0. \tag{A-2}$$

Let us define two other functions, $u(x)$ and $v(x)$, such that

$$u(x) = \frac{\frac{d}{dx} f(x + \alpha) - \frac{d}{dx} f(x)}{f(x + \alpha) - f(x)} \tag{A-3}$$

and

$$v(x) = \frac{\frac{d^2}{dx^2} f(x)}{\frac{d}{dx} f(x)} \tag{A-4}$$

where α is any finite positive number.

If $v(x)$ is strictly decreasing in x, or we have

$$\frac{d}{dx} v(x) \leq 0, \tag{A-5}$$

where an equality holds at most at an enumerable number of points of x, then $u(x)$ is also strictly decreasing in x, or we have

$$\frac{d}{dx} u(x) \leq 0, \tag{A-6}$$

where an equality holds at most at an enumerable number of points of x.

Proof:

Let us define $g(x)$ and $h(x)$ such that

$$g(x) = \frac{d}{dx} f(x + \epsilon) - \frac{d}{dx} f(x) = \int_x^{x+\epsilon} \frac{d^2}{dt^2} f(t)\, dt \tag{A-7}$$

and

$$h(x) = f(x + \epsilon) - f(x) \tag{A-8}$$

$$= \int_x^{x+\epsilon} \frac{d}{dt} f(t)\, dt$$

where ϵ is any positive number.

Since we have (A-2), by Cauchy's mean value theorem we can write

$$v(\zeta) = \frac{g(x)}{h(x)} \tag{A-9}$$

for some value ζ satisfying

$$x < \zeta < x + \epsilon, \tag{A-10}$$

and also

$$v(\xi) = \frac{g(x - \epsilon)}{h(x - \epsilon)} \tag{A-11}$$

for some value ξ satisfying

$$x - \epsilon < \xi < x. \tag{A-12}$$

From this and (A-5) we can write

$$\begin{aligned} v(x - \epsilon) &> \frac{g(x - \epsilon)}{h(x - \epsilon)} > v(x) \\ &> \frac{g(x)}{h(x)} > v(x + \epsilon). \end{aligned} \tag{A-13}$$

Replacing $(x + s\epsilon)$ for x in (A-13) where s is an integer, we have

$$\begin{aligned} v(x + (s - 1)\epsilon) &> \frac{g(x + (s - 1)\epsilon)}{h(x + (s - 1)\epsilon)} > v(x + s\epsilon) \\ &> \frac{g(x + s\epsilon)}{h(x + s\epsilon)} > v[x + (s + 1)\epsilon]. \end{aligned} \tag{A-14}$$

On the other hand, from (A-3), (A-7) and (A-8) we have

$$\begin{aligned} u(x) &= \frac{\displaystyle\int_x^{x+\alpha} \frac{d^2}{dt^2} f(t)\, dt}{\displaystyle\int_x^{x+\alpha} \frac{d}{dt} f(t)\, dt} \\ &= \frac{\displaystyle\sum_{r=0}^{m-1} g(x + r\epsilon)}{\displaystyle\sum_{r=0}^{m-1} h(x + r\epsilon)} \end{aligned} \tag{A-15}$$

by setting ϵ and m so that they should satisfy

$$\epsilon = \frac{\alpha}{m}. \tag{A-16}$$

In a similar manner we can write

$$\text{(A-17)} \qquad u(x + \epsilon) = \frac{\sum_{r=1}^{m} g(x + r\epsilon)}{\sum_{r=1}^{m} h(x + r\epsilon)} .$$

From (A-2) and (A-8) it is obvious that $h(x)$ is positive, and then we have from (A-14)

$$\text{(A-18)} \qquad \frac{g(x)}{h(x)} > \frac{\sum_{r=1}^{m-1} g(x + r\epsilon)}{\sum_{r=1}^{m-1} h(x + r\epsilon)} > \frac{g(x + \alpha)}{h(x + \alpha)} ,$$

and from this and (A-15) and (A-17) we have

$$\text{(A-19)} \qquad u(x) > \frac{\sum_{r=1}^{m-1} g(x + r\epsilon)}{\sum_{r=1}^{m-1} h(x + r\epsilon)} > u(x + \epsilon).$$

Since ϵ can be as small as we wish, from (A-19) we can finally conclude that $u(x)$ is strictly decreasing in x. Thus the proof has been completed.

It should be noted that, if we define $u^*(x)$ in such a manner that

$$\text{(A-3)*} \qquad u^*(x) = \frac{\frac{d}{dx} f(x) - \frac{d}{dx} f(x - \alpha)}{f(x) - f(x - \alpha)}$$

instead of $u(x)$ in (A-3), the same conclusion will be reached. For, if we define variable y such that

$$\text{(A-20)} \qquad y = x + \alpha,$$

we can rewrite (A-3) and (A-4) in the forms

$$\text{(A-21)} \qquad u(x) = u^*(y) = \frac{\frac{d}{dy} f(y) - \frac{d}{dy} f(y - \alpha)}{f(y) - f(y - \alpha)}$$

and

$$\text{(A-22)} \qquad v(x + \alpha) = v(y) = \frac{\frac{d^2}{dy^2} f(y)}{\frac{d}{dy} f(y)} ,$$

and the range of y is given by

$$\text{(A-23)} \qquad -\infty < y < \infty .$$

In a special case where $f(x)$ has an upper asymptote, $\bar{C}(< \infty)$, we have

$$\lim_{x \to \infty} f(x) = \bar{C} \tag{A-24}$$

and

$$\lim_{x \to \infty} \frac{d}{dx} f(x) = 0. \tag{A-25}$$

From this we can easily see that, if α tends to positive infinity, $f(x + \alpha)$ tends to $\bar{C}$, and its first derivative tends to zero. Defining $u^{**}(x)$ such that

$$u^{**}(x) = \lim_{\alpha \to \infty} u(x) = \frac{-\frac{d}{dx} f(x)}{\bar{C} - f(x)}, \tag{A-26}$$

we can prove that, if (A-5) is true, we obtain

$$\frac{d}{dx} u^{**}(x) \leq 0, \tag{A-27}$$

where an equality holds at most at an enumerable number of points of x. For, by following a similar logic as before, we can write

$$u^{**}(x) = \frac{\sum_{r=0}^{\infty} g(x + r\epsilon)}{\sum_{r=0}^{\infty} h(x + r\epsilon)} \tag{A-28}$$

and

$$u^{**}(x + \epsilon) = \frac{\sum_{r=1}^{\infty} g(x + r\epsilon)}{\sum_{r=1}^{\infty} h(x + r\epsilon)}, \tag{A-29}$$

and also

$$\frac{g(x)}{h(x)} > \frac{\sum_{r=1}^{\infty} g(x + r\epsilon)}{\sum_{r=1}^{\infty} h(x + r\epsilon)}. \tag{A-30}$$

Thus from (A-28), (A-29) and (A-30) we obtain

$$u^{**}(x) > \frac{\sum_{r=1}^{\infty} g(x + r\epsilon)}{\sum_{r=1}^{\infty} h(x + r\epsilon)} = u^{**}(x + \epsilon), \tag{A-31}$$

and the proof has been completed.

In another special case where $f(x)$ has a lower asymptote, $\underline{C}(> -\infty)$, we can write

(A-32) $$\lim_{x\to-\infty} f(x) = \underline{C}$$

and

(A-33) $$\lim_{x\to-\infty} \frac{d}{dx} f(x) = 0.$$

From this fact it can be seen that, if α tends to positive infinity, $f(x - \alpha)$ tends to $\underline{C}$, and its derivative tends to zero. Using (A-3)* instead of (A-3) and defining $u^{***}(x)$ such that

(A-34) $$u^{***}(x) = \lim_{\alpha\to\infty} u^{*}(x) = \frac{\frac{d}{dx} f(x)}{f(x) - \underline{C}},$$

we can see that, if (A-5) is true, we obtain

(A-35) $$\frac{d}{dx} u^{***}(x) \leq 0.$$

where an equality holds at most at an enumerable number of points of x. For, in the same way as before, we reach

(A-36) $$u^{***}(x) = \frac{\sum_{r=1}^{\infty} g(x - r\epsilon)}{\sum_{r=1}^{\infty} h(x - r\epsilon)}$$

and

(A-37) $$u^{***}(x + \epsilon) = \frac{\sum_{r=0}^{\infty} g(x - r\epsilon)}{\sum_{r=0}^{\infty} h(x - r\epsilon)},$$

and also

(A-38) $$\frac{\sum_{r=1}^{\infty} g(x - r\epsilon)}{\sum_{r=1}^{\infty} h(x - r\epsilon)} > \frac{g(x)}{h(x)}.$$

Thus from (A-36), (A-37) and (A-38) we obtain

$$u^{***}(x) = \frac{\sum_{r=1}^{\infty} g(x - r\epsilon)}{\sum_{r=1}^{\infty} h(x - r\epsilon)} > u^{***}(x + \epsilon), \tag{A-39}$$

and the proof has been completed.

If $f(x)$ has unity and zero as its upper and lower asymptotes, we have from (A-26) and (A-34)

$$u^{**}(x) = \frac{-\frac{d}{dx} f(x)}{1 - f(x)} \tag{A-40}$$

and

$$u^{***}(x) = \frac{\frac{d}{dx} f(x)}{f(x)} \cdot \tag{A-41}$$

It is obvious from (A-3), or (A-3)*, and (A-4) that

$$\lim_{\alpha \to 0} u(x) = v(x). \tag{A-42}$$

From this fact we can conclude that, for (A-6) to be true for *any* positive number α, (A-5) is the necessary and sufficient condition.