PSYCHOMETRIC MONOGRAPH NUMBER 14

AN ANALYTICAL AND EMPIRICAL EXAMINATION OF SOME PROPERTIES OF IPSATIVE MEASURES

This Study Was Supported in Part by
Office of Naval Research Contract Nonr-477(08)
and
Public Health Research Grant M-743(C2).

The William Byrd Press, Inc.
Richmond, Va.

AN ANALYTICAL AND EMPIRICAL EXAMINATION OF SOME PROPERTIES OF IPSATIVE MEASURES

William V. Clemans

Science Research Associates, Inc.

This monograph in a modified form was submitted in partial fulfillment of the requirements for a Ph.D. degree at the University of Washington in October 1956. The writer wishes to thank his sponsor, Paul Horst, for advice and counsel related to the study.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

A frequently considered problem, in fact one listed as a fundamental question by Thomson [15], concerns what metric or system of units is to be used in factorial analysis. Although it is not so frequently considered in some of the other areas of quantitative analysis, the problem of metric is always there and always of fundamental importance. Some researchers have become aware of a facet of the metric problem only after the discouraging experience of attempting to find the inverse of a correlation matrix when no inverse existed because of a property introduced by the metric utilized.

The general problem of metric in psychological measurement is far too extensive to be covered in this monograph. Here only the properties of a certain class of units will be examined.

General Statement of Problem

Several years ago Cattell [3] stated that the psychological measurement of behavior could be expressed in three kinds of units: (i) "raw" or "interactive" units which are neither dependent on any other scores of the individual measured nor upon the scores of any other individuals, (ii) "normative" units where the score of the individual is dependent upon the scores of other individuals in the population, and (iii) "ipsative" units where each score for an individual is dependent on his scores on other variables. Raw units are the most familiar; they are used in all fields of science as well as in many non-science areas. Their properties are well known and their usefulness is a matter of common accord. Normative units have been used for many years especially in the social sciences. A factor which no doubt contributes to the frequency use of normative units in psychology is the existence of many measurement problems where it is extremely difficult to establish an adequate zero point. Normative units are recognized as very useful in psychological measurement, and in general their properties are quite well known.

The properties of ipsative units are not well known. The purpose of this monograph is to examine both from a theoretical and an empirical point of view the properties of such units. The usefulness of ipsative measures will not be questioned. Rather it will be accepted that instruments yielding ipsative scores have been found to be useful, and, as a consequence, are being administered with increasing frequency. It seems, therefore, important and timely to examine some of their properties.

Specific Aims of Study

This study is designed (i) to examine the relation of normative scores based on raw units to normative scores based on ipsative units; (ii) to examine the changes in the intercorrelations of a set of variables and their validity coefficients occasioned by the conversion of the primary data from raw to ipsative units; (iii) to determine analytically and empirically the relation between the multiple correlation of a criterion with a set of variables in raw score units and the multiple correlation of the same criterion with the same set of variables in ipsative units; (iv) to compare the predictive efficiency of a raw score matrix when a variable is deleted with the predictive efficiency of an ipsative score matrix of the original variables with none deleted; (v) to consider the influence of the change from raw to ipsative scores on the multiple correlation with a specified criterion when certain assumptions—such as initially assuming complete independence of predictor variables—are made with respect to the intercorrelations and validity coefficients of the variables when in raw form; (vi) to demonstrate analytically and empirically that the least square solution for predicting a criterion using all of the variables of an ipsative set is identical with the least-square solution with any single variable of the ipsative set removed; (vii) to determine in a more analytical manner than has yet been done why certain observed properties of ipsative scores hold (for example, why are more than half of the elements in ipsative correlation matrices negative?); (viii) to examine the relation between the first-centroid residual of the intercorrelation matrix of a set of variables and the intercorrelation matrix for the same variables after conversion to ipsative units.

Related Studies

The literature contains few studies closely related to the topics covered here.* In Cattell's article [3], calling attention to the differences among raw, normative, and ipsative measurements, only fifteen references are cited, and only one of these, Thomson [14], includes material relevant to this study. Without presenting any mathematical evidence, Cattell concluded that, "... One may point out that the value of knowing whether a measurement is ipsative, normative, or interactive is that the knowledge often safeguards the experimenters against improper manipulation or interpretation of the measurements" ([3], p. 302).

Cattell seems to imply that most experimenters are well aware of the characteristics of these three types of measurement, but Thomson [15] gives an example which suggests that even Cyril Burt did not have a clear understanding of the properties of ipsative variables. In his chapter on the relation between test factors and person factors, Thomson cites an example designed

* Drawing on a prepublication copy of this study, J. A. Radcliffe derived and restated some of the findings in the *Australian Journal of Psychology,* 15, No. 1, 1963.

by Burt to show that, if the initial units are suitably chosen, the factors of the one kind of analysis are identical with the loadings of the other, and vice versa. Thomson states that he, "while agreeing that this is so in the very special circumstances assumed by Burt, is of opinion that his is a very narrow case, and that the factors considered by Burt are not typical of those in actual use in experimental psychology" ([15], p. 263).

Guilford [6] in 1952 indicated that ipsative variables should not be used in standard factor analytic procedures. The reasons advanced, however, were not analytical. He pointed out that the correlation matrices based on such variables as those obtained using the Kuder, a partially ipsative test, have about two-thirds of their elements negative. Harris [8] attempted, though, not too successfully, to relate the factors obtained from ipsative variables to the set of factors that would have been obtained from the same variables in non-ipsative units.

Although some observations will be made concerning the rank of an ipsative score matrix, it is not a major purpose of this study to examine the relation of ipsative variables to factor analytic procedures. The above quotation from Thomson is also only indirectly related to this study and was given to add weight to the statement made earlier that the properties of ipsative measurements are not well known.

CHAPTER II

DEVELOPMENT OF IPSATIVE MATRIX

The purpose of this chapter is to make the concept of an ipsative variable more meaningful to those unfamiliar with the term. The relation of ipsative matrices to primary data, normative, and ipsative-standard matrices will also be discussed. Contrived examples will be used for this purpose and for the purpose of introducing some of the properties of ipsative units.

Definition of Ipsative Score Matrix

The term *ipsative* was suggested by Cattell [3] because it seemed a convenient one for designating scales in which the units are relative to other measurements on the person himself. Here a comparable but more mathmatical definition suggested by Horst (personal communication) has been used: any score matrix, which has the property that the sum of the scores over the attributes for each of the entities is a constant, will be said to be ipsative. The general term *entity* will be used throughout to designate any organism or thing with measurable attributes.

Primary Data Matrix

Any set of scores can be made ipsative by simply adding a suitable constant to the measure of each attribute for a specific entity such that the new scores will sum to the same constant for all entities. However, unless precautions are taken to insure that the variables are in standard units before "ipsatizing" the resulting scores will be devoid of meaning. This statement will perhaps be better understood by reference to Table 1 which gives fictitious primary data or raw scores for ten subjects or entities on four attributes.

Table 1 clearly shows that the units of measurement for the four attributes vary in the extreme both with respect to means and standard deviations. Attribute I measures could be thought of as scores on a qualifying examination; Attribute II measures, as undergraduate grade point averages; Attribute III measures, as scores on a general adjustment inventory; and Attribute IV, as a measure of educational progression. In this instance Attribute IV has a constant value for each of the entities, as they are all first-year graduate students.

The primary data matrix in Table 1 could be made ipsative by adding a suitable constant to the scores in each row, but the resulting scores would be essentially nonsense units. It would appear, for instance, that each individual possessed much more of Attribute I than of any of the other at-

TABLE 1

Primary Matrix of Raw Scores

Entity	Attribute I	II	III	IV
1	700	3.8	34	5
2	600	4.0	33	5
3	650	3.6	31	5
4	550	3.1	22	5
5	400	2.8	12	5
6	500	3.2	22	5
7	450	2.8	15	5
8	350	2.7	11	5
9	300	2.5	6	5
10	500	2.5	14	5
M	500	3.1	20	5
σ	129.10	.54	9.98	0

tributes. Before ipsative scores can be meaningfully obtained it is essential to transform the measures for each attribute into some standard form; otherwise the resulting units would be what Cattell has called ". . . a bastard ipsative measure, formed by putting together interactive measurements of quite different modality" ([3], p. 296).

Standard Score Matrix

As a preliminary measure prior to ipsatizing, the scores in Table 1 were transformed into T-scores with a mean of 50 and a standard deviation of 10 except for Attribute IV which, as it was a constant, simply had 50 substituted throughout. These scores are given in Table 2. It should be noted that even with this common standardizing procedure some valuable information may be lost due to the arbitrary equating of means and variances. This is true because there certainly is something to be said for the probability that real differences of standard deviation exist between tests. Equating the standard deviations is, as Thomson has stated, ". . . in a sense a confession of ignorance" ([15], p. 329).

Although Table 1 certainly contains more information than is available in Table 2, the information in the latter table is more immediate. By direct observation the column order in Table 1 was meaningful, but the row order was meaningless: that is, differences between scores in the columns represent behavioral differences, but in no interpretable sense can the differences between scores in the rows be treated as behavioral differences. In Table 2 the relationship among scores in the columns remains the same, and the differences between scores in the rows have meaning in that they can be compared with respect to how they rank the individual in the group. For

TABLE 2

Standard Scores

Entity	Attribute I	II	III	IV	Σ
1	65	63	64	50	242
2	58	67	63	50	238
3	62	59	61	50	232
4	54	50	52	50	206
5	42	44	42	50	178
6	50	52	52	50	204
7	46	44	45	50	185
8	38	43	41	50	172
9	35	39	36	50	160
10	50	39	44	50	183
M	50	50	50	50	
σ	10.0	10.0	10.0	10.0	

example, the scores in the tenth row directly indicate that Entity 10 is at the mean on Attribute I and a little more than a standard deviation below the group mean on Attribute II. Hence, with respect to the group, it is meaningful to say that his score on Attribute I is higher than on Attribute II. This judgment could not have been made by looking only at the tenth row in Table 1.

Ipsative Score Matrix

If one is only interested in the information contained in the rows and is not concerned with the loss of the information contained in the columns, Table 2 can be transformed into a still different form. This form is the *ipsative* score matrix which emphasizes intra-individual differences, and changes inter-individual relationships. The scores in Table 2 were transformed into ipsative units by adding a constant to the entries in each row such that the new scores for each entity given in Table 3 sum to 100 within rounding error.

The constant to be added to each row can be readily determined by first summing the row, then subtracting this sum from 100, and finally dividing by the number of variables, in this case four. For example, the sum of the first row in Table 2 is 242; by subtracting this value from 100 and dividing by 4, the constant, −35.5, is obtained. This is the constant which when added to the values in the first row of Table 2 yields the entries in the first row of Table 3.

Note that the means of the attributes in Table 3 are identical. This must be true if the scores previous to ipsatizing had identical means. (For proof see Chap. IV.) However, the standard deviations of the attributes are not equal and appear unrelated to the standard deviations found in Table 1.

TABLE 3

Ipsative Scores

Entity	Attribute I	II	III	IV	Σ
1	29.5	27.5	28.5	14.5	100.0
2	23.5	32.5	28.5	15.5	100.0
3	29.0	26.0	28.0	17.0	100.0
4	27.5	23.5	25.5	23.5	100.0
5	22.5	24.5	22.5	30.5	100.0
6	24.0	26.0	26.0	24.0	100.0
7	24.7	22.7	23.7	28.7	99.8
8	20.0	25.0	23.0	32.0	100.0
9	20.0	24.0	21.0	35.0	100.0
10	29.2	18.2	23.2	29.2	99.8
M	24.99	24.99	24.99	24.99	
σ	3.65	3.64	2.71	7.29	

Note, for example, that Attribute IV which was originally a constant is now, when in ipsative form, the measure with the greatest degree of variability. The resulting variability of such an originally constant score is a function of the variance of the row sums of the original score matrix.

Ipsative-Standard Score Matrix

The relative values of strengths of the attributes of an individual when compared with each other can be obtained by ranking his ipsative scores, providing the means and variances of the attributes were equated prior to ipsatizing. If additional information is desired relative to how an individual's ranking of an attribute compares with the ranking of the same trait by others, the ipsative scores for a given attribute can be transformed into standard scores by columns. These ipsative-standard scores are given in Table 4.

The argument was presented above in reference to Table 1 that if the means and the variances were not equated prior to ipsatizing then rank ordering an individual's scores will have no meaning. However, if the assumption can be made that the variances were equated prior to ipsatizing and that only the means were different, then it is still possible to determine the rank ordering of the relative value or strength of an individual's scores. However, under these circumstances it is *absolutely essential* to transform the ipsative scores into deviation units for each attribute before the relative values or strengths of an individual's scores in relation to each other can be determined. The results obtained under these conditions will be identical to those that would have been obtained had the means been equated prior to ipsatizing.

TABLE 4

Ipsative-Standard Scores

Entity	Attribute I	II	III	IV
1	62	57	63	36
2	46	71	63	37
3	61	53	61	39
4	57	46	52	48
5	43	49	41	58
6	47	53	54	49
7	49	44	45	55
8	36	50	43	60
9	36	47	35	64
10	62	31	43	56
M	49.9	50.1	50.0	50.2
σ	10.2	10.3	10.0	10.0

If the standardization group for any scale that gives ipsative scores directly yields means that are different, then the above transformation to deviation scores—not standard scores— must be made before an individual's scores can be ranked. Inasmuch as the primary purpose of ipsative scores is to make intra-individual comparisons, it is of the utmost importance for users of ipsative variables to be aware of this necessity.

In the event that the raw score variances underlying a set of ipsative scores were unequal, then the resulting ipsative scores are difficult if not impossible to interpret and no transformation will restore the information lost.

Effect on Interpretation Occasioned by Ipsative Transformations

Each transformation of the variables makes a somewhat different interpretation of an individual's scores possible and in fact sometimes necessary. In Table 5 the scores on the four attributes for individuals 2 and 10 are given in each of the four types of units discussed previously in this chapter.

When the scores are in raw units there is no problem in telling whether individual 2 or 10 has the higher score for a given attribute. However, the scale locations of the scores of individuals 2 and 10 are not obvious, that is, it is unclear whether their scores are above or below the mean and how much distance is represented by the difference between any two scores. Furthermore, when the scale locations are unknown there is no basis for comparing the scores for an individual on different attributes; that is, more information is needed before it can be determined whether individual 10 ranks higher on Attribute II or III.

When the scores are in standard units, it not only is possible to tell

TABLE 5

Four Types of Attribute Scores for Two Subjects

	Raw Scores				Standard Scores			
	Attribute				Attribute			
S	I	II	III	IV	I	II	III	IV
2	600	4.0	33	5	58	67	63	50
10	500	2.5	14	5	50	39	44	50
	Ipsative Scores				Ipsative-Standard Scores			
	Attribute				Attribute			
S	I	II	III	IV	I	II	III	IV
2	23.5	32.5	28.5	15.5	46	71	63	37
10	29.2	18.2	23.2	29.2	62	31	43	56

immediately whether individual 2 or 10 has the higher score for a given attribute, but the location of this score on the scale of the given attribute with respect to some base group is made clear. For example, the score for individual 10 on Attribute I places him right at the mean and individual 2's score is eight tenths of a standard deviation above him. It is also now possible to compare the scores for an individual on the different attributes. For instance, individual 10 has equivalent scores (with respect to the group) for Attributes I and IV, being at the mean on both; he is below the mean on Attribute III and his lowest score is on Attribute II.

When the scores are in ipsative units, it is no longer possible to determine which of a number of individuals has the most of a specified attribute (i.e., scores the highest). The comparisons between individuals are now limited to rank comparisons. Note the raw and standard scores for individual's 2 and 10 on Attribute I. It is clear that individual 2 possesses more of Attribute I than does individual 10. But this order is just reversed when the scores are transformed into ipsative units. This is true because individual's 2 scores on Attributes II and III are higher than his score on Attribute I and the situation is just reversed for individual 10. As a consequence the ranking of individual 2's score on Attribute I, relative to his other scores, is lower than the ranking of individual 10 on Attribute I relative to his other scores. The intra-individual comparisons are exactly the same as when the scores were in standard units.

When the scores are in ipsative-standard units, it is possible to tell easily how an individual's ranking of an attribute compares with the ranking of the attribute by others. However, standardizing the ipsative scores for each attribute sometimes makes it difficult or impossible to make score comparisons between the attributes for a given individual. For example, when in either standard or ipsative score form, it was clear that individual 10 had equivalent scores for Attributes I and IV. When in ipsative-standard form, however, it would appear that he ranks higher on Attribute I than on IV. This phenomenon occurred because the variances of the ipsative variables were not identical.

For a review of the effects of the various score transformations, consider column comparisons as those between individuals for an attribute and consider row comparisons as those between attributes for an individual. When the scores are in raw units, column comparisons are meaningful, but row comparisons are meaningless. In standard score units both column and row comparisons are meaningful, and locations of scores on the scale are obvious. In ipsative units the row order is meaningful, but column order is meaningless as far as the absolute strength of a given attribute is concerned; however, it is possible within the column to compare the ranking of an attribute for different individuals. When in ipsative-standard units both row and column comparisons are meaningless as far as the absolute strengths of the attributes are concerned. Although the scores within the column can be used to compare the relative ranking of an attribute for different individuals, these scores cannot be used to show that individual 2's score, for example, is stronger in absolute strength than individual 10's.

Effect of Deleting an Attribute Prior to Ipsatizing

The example carried along from Table 1 through Table 5 was contrived to demonstrate among other things the effect resulting from the inclusion of a constant with the variables to be ipsatized. The results are interesting, but perhaps not very realistic because no one would knowingly include a constant in a set of variables to be ipsatized. However, it should be noted that when the scores are originally determined in ipsative form, as in the Allport-Vernon *Study of Values,* the underlying or raw variances are unknown and cannot be estimated from the ipsative data. Hence, it is not impossible that what appears as an *ipsative variable* may stem from an underlying *raw-score constant.*

Because the inclusion of a constant in the original set of raw scores is probably atypical, another example of an ipsative set of data was obtained by ipsatizing only the first three attributes in Table 1. This new set of ipsative units is given in Table 6 along with the corresponding set of ipsative-standard scores.

It can be noted from Tables 3 and 6 that the order within corresponding columns is not the same. The actual correlations between the three corresponding variables of each set are far from perfect. They are, in order, .71, .68, and .34. It can also be noted from observing the two tables that the standard

TABLE 6

Scores Obtained by Transforming First Three Attribute Measures in Table 1

	Ipsative Scores				Ipsative-Standard Scores		
	Attribute				Attribute		
Entity	I	II	III	Σ	I	II	III
1	34.3	32.3	33.3	99.9	53	46	49
2	28.7	37.7	33.7	100.1	34	66	58
3	34.7	31.7	33.7	100.1	55	44	58
4	35.3	31.3	33.3	99.9	57	43	49
5	32.7	34.7	32.7	100.1	48	55	35
6	32.0	34.0	34.0	100.0	45	52	65
7	34.3	32.3	33.3	99.9	53	46	49
8	30.7	35.7	33.7	100.1	41	58	58
9	31.7	35.7	32.7	100.1	44	58	35
10	39.0	28.0	33.0	100.0	70	31	42
M	33.34	33.34	33.34		50.0	49.9	49.8
σ	2.85	2.79	.44		10.0	9.9	10.2

deviations for corresponding columns have different values. However, the row order of the ipsative scores remains unchanged.

The corresponding columns of the ipsative-standard scores of Tables 4 and 6 of course correlate to the same degree with each other as the pairs of prenormalized ipsative variables. Now, however, the *row* order has been altered. For example, the ipsative-standard scores for individual 8 were 36, 50, and 43 in Table 4 whereas in Table 6 they are respectively 41, 58, and 58.

This example makes it clear that altering the number of variables in a set will sometimes alter the order within pairs of corresponding columns. It also makes it clear that the row order is not changed until the transformation to ipsative-standard units is made. Of course when ipsative scores are used it is the *row* order that is of prime importance. Hence, as it would usually be desirable to keep this order independent of the number of variables, it would be advantageous for interpretive purposes to use the scores in nonnormalized form.

Ipsatizing Under Conditions of Unequal Variance

It was implied above that the most essential prerequisite to ipsatizing was equivalent variances for all of the attributes in the set. It is interesting to note what occurs when this condition is not met. To show by example the effect of ipsatizing a set of variables with unequal variances, the first

TABLE 7

Effect of Ipsatizing on Primary Matrix Having Attribute Measures with Unequal Variances

Entity	Original Scores with Modified Variances			Ipsative Scores			Ipsative-Standard Scores		
	Attribute			Attribute			Attribute		
	I	II	III	I	II	III	I	II	III
1	65	76	57.0	32.3	43.3	24.3	47	61	36
2	58	84	56.5	25.2	51.2	23.7	30	69	35
3	62	68	55.5	33.5	39.5	27.0	50	57	40
4	54	50	51.0	35.7	31.7	32.7	56	48	49
5	42	38	46.0	33.3	29.3	37.3	50	46	56
6	50	54	51.0	31.7	35.7	32.7	46	53	49
7	46	38	47.5	35.5	27.5	37.0	55	44	56
8	38	36	45.5	31.5	29.5	39.0	46	46	59
9	35	28	43.0	33.0	26.0	41.0	49	42	62
10	50	28	47.0	21.7	19.7	38.7	70	35	58
M	50.0	50.0	50.0	33.34	33.34	33.34	49.9	50.1	50.0
σ	10.0	20.1	5.0	4.13	9.25	6.36	10.0	10.0	9.9

three variables in Table 2 were used except that the standard deviation was doubled for Attribute II and halved for Attribute III while the means were kept constant. The resulting scores were then ipsatized and finally normalized. The three sets of scores are found in Table 7.

A comparison of the ipsative-standard scores in Table 6 and Table 7 reveals that except for Attribute I, the attribute with the unchanged variance, there is little similarity between the two sets of data. The correlation with Attribute III in Table 6 with its counterpart in Table 7 is actually negative. The correlations between the corresponding ipsative variables in Tables 6 and 7 are, in order, .93, .47, and −.57.

Summary

This chapter defined and gave a brief introduction to ipsative scores. A set of attribute measures was defined as ipsative when the score sum over all attributes for each entity is constant. Three sets of ipsative scores were calculated from a fictitious set of primary data. Several observations, such as the necessity for standardizing prior to ipsatizing, were made on ipsative matrices in general. Observations on the specific examples showed: (i) the effect on score interpretation necessitated by the transformation from one type of score to another, (ii) the effect of deleting an attribute prior to ipsatizing, and (iii) the effect of failing to equate the standard deviations of the attributes prior to ipsatizing.

CHAPTER III

AN EMPIRICAL EXAMPLE

In Chapter II a set of fictitious primary data was utilized to introduce the concept of an *ipsative* score matrix. The example to be developed in this chapter is based on an actual sample of 129 students at the University of Washington. As in the contrived example of Chapter II, the primary data are in terms of *raw* units. Although some scales, such as *A Study of Values*, yield scores directly in ipsative units, it was necessary for the purposes of this study to use raw scores because they can always be transformed into ipsative units, whereas there is insufficient information to go from ipsative units to raw scores. This is somewhat analogous to the observation that raw scores can always be transformed into standard units, but unless the original mean and standard deviation are known it is not possible to transform standard scores into raw units.

Raw scores are essential here because several of the aims of this study, listed in Chapter I, call for a comparison between raw and ipsative scores. The measurement of the absolute value or magnitude of a trait is only possible with raw units. Admittedly, raw scores are not necessarily valid measures of the absolute value or magnitude of a trait, but they are possible measures. On the other hand ipsative scores, by definition, are *relative* measures, never *absolute* measures—something that should always be remembered when using scales yielding ipsative scores. Many years before Cattell coined the term ipsative, Allport and Vernon in their manual for *A Study of Values* clearly stated the reservations which must be kept in mind when interpreting "ipsative" scores. They said:

> *The test measures only the relative strength of the six evaluative attitudes.* A high score in one value can be obtained only by reducing correspondingly the scores on one or more of the other values. In interpreting the results, therefore, it is necessary to bear in mind that they reveal only the *relative* importance of each of the six values in a given personality, not the total amount of "value energy" or drive possessed by an individual. It is quite possible for the highest value of a generally apathetic person to be less intense and effective than the lowest value of a person in whom all values are prominent and dynamic ([1], p. 6).

Sample and Data for Empirical Example

The 129 students making up the sample for this study entered the University of Washington in the Fall of 1953. Six predictor scores and a criterion score were obtained for each student. The six predictors were: (1) Guilford-Zimmerman I, a vocabulary test;, (2) Guilford-Zimmerman VII,

TABLE 8

Intercorrelations and Validity Coefficients with Grade Point Average for Six Predictor Variables

Predictor	Intercorrelations 1	2	3	4	5	6	Validity Coefficients
1	1.000	.004	.546	.344	.572	.349	.487
2	.004	1.000	−.178	.505	.203	.243	−.018
3	.546	−.178	1.000	.207	.356	.364	.328
4	.344	.505	.207	1.000	.496	.595	.309
5	.572	.203	.356	.496	1.000	.493	.529
6	.349	.243	.364	.595	.493	1.000	.309
Σ	2.815	1.777	2.295	3.147	3.120	3.044	1.944

a mechanical knowledge test; (3) Cooperative English Test, spelling section only; (4) Cooperative Mathematics Test, Part I only; (5) Cooperative Social Studies Test, Part II only; (6) American Council on Education Psychological Examination, Quantitative Scale only. The criterion score was the student's cumulative grade point average for the first two years of university study. The complete set of raw scores is given in the Appendix. Table 8 gives the matrix of intercorrelations of the predictor variables and the vector of validity coefficients.

The examples in Chapter II served to demonstrate the differences between raw scores and ipsative scores as defined in Chapter I. For this reason and because it is possible to compute the ipsative intercorrelation matrix as a function of the raw score intercorrelation matrix, the ipsative scores were not computed for the sample of 129 students. What follows will indicate how this can be done. (Derivations of formulas in this chapter will be given in Chapter IV.)

Ipsative Covariance and Correlation Matrices

Without loss of generality, it will be assumed in the mathematical derivations that the original scores were transformed into normal units with a mean of zero and a standard deviation of one prior to ipsatizing. Under these circumstances the ipsative covariance matrix is given by

$$C = \left(I - \frac{1\,1'}{m}\right)r\left(I - \frac{1\,1'}{m}\right), \tag{1}$$

where

C is the ipsative covariance matrix,
I is the identity matrix,

r is the intercorrelation matrix for the original scores,
m is the number of predictors,
1 is a vector with all unit elements.

The ipsative covariance matrix is given in Table 9. Note that the sums of the columns (or rows) are equal to zero within rounding error. This will always be true of ipsative covariance matrices regardless of the original unit of measurement.

The reader familiar with matrix notation will of course be aware that the elements in the principal diagonal of C are the variances of the ipsative variables for the case when the variables prior to ipsatizing were in normal units with a mean of zero and a standard deviation of one. As stated above the original variances must all be equated before ipsatizing, but whatever the value of the equated variances, it can be shown that the relative magnitude of the resulting ipsative variances will be dependent on the intercorrelations of the original variables.

Once the ipsative covariance matrix is at hand the ipsative correlation matrix can be readily obtained by

$$(2) \qquad {}_i r = D_C^{-1/2} C D_C^{-1/2},$$

where $D_C^{-1/2}$ is a diagonal matrix of the reciprocals of the square roots of the principal diagonal elements in C. Equations (1) and (2) show that it is possible to obtain the ipsative intercorrelation matrix as a function only of the number of variables and the original intercorrelations. This is always true whether or not the original scores were in normal units with a mean of zero and a standard deviation of unity as long as it is known that the underlying variances were equated prior to ipsatizing. Starting with the intercorrelations of the first three attribute measures given in Table 1 of Chapter II, it is possible to demonstrate the correspondence between the intercorrelations actually calculated from the ipsative scores and the ipsative inter-

TABLE 9

C Matrix of Ipsative Covariances of Six Predictor Variables

Predictor	1	2	3	4	5	6	Σ
1	.5116	−.3114	.1443	−.1998	.0327	−.1776	−.0002
2	−.3114	.8576	−.4067	.1342	−.1633	−.1106	−.0002
3	.1443	−.4067	.6850	−.2501	−.0966	−.0759	.0000
4	−.1997	.1343	−.2500	.4009	−.0986	.0131	.0000
5	.0328	−.1632	−.0965	−.0986	.4099	−.0844	.0000
6	−.1775	−.1105	−.0758	.0131	−.0844	.4353	.0002
Σ	.0001	.0001	.0003	−.0003	−.0003	−.0001	−.0002

TABLE 10

Original Intercorrelation Matrix for First Three Variables of Table 1 with Derived and Calculated Ipsative Intercorrelation Matrices

Variable	Original Intercorrelation Matrix			Ipsative Intercorrelation Matrices					
				Derived from Formula (2)			Calculated from Scores in Table 7		
	I	II	III	I	II	III	I	II	III
I	1.000	.845	.953	1.00	.99	−.34	1.00	.99	−.32
II	.845	1.000	.946	.99	1.00	.18	.99	1.00	.17
III	.953	.946	1.000	−.34	.18	1.00	−.32	.17	1.00

correlation matrix obtained as a function of the original intercorrelations. Table 10 gives all three matrices. The discrepancies between the two ipsative matrices are due only to an accumulation of rounding errors.

The ipsative intercorrelation matrix for the empirical example is given in Table 11. It was obtained by using Formula (2).

The validity coefficients for a set of ipsative variables are also readily obtained if the original validity coefficients are known. The relationship is given by

$$(3) \qquad {}_ir_c = D_C^{-1/2}\left(I - \frac{1\,1'}{m}\right)r_c,$$

where

D_C and m are as defined previously,
${}_ir_c$ is the vector of ipsative validity coefficients,
r_c is the vector of original validity coefficients.

TABLE 11

Ipsative Intercorrelation Matrix of Six Predictor Variables

Predictor	1	2	3	4	5	6
1	1.000	−.470	.244	−.441	.072	−.376
2	−.470	1.000	−.531	.229	−.275	−.181
3	.244	−.531	1.000	−.477	−.182	−.139
4	−.441	.229	−.477	1.000	−.243	.031
5	.072	−.275	−.182	−.243	1.000	−.200
6	−.376	−.181	−.139	.031	−.200	1.000
Σ	.029	−.228	−.085	.099	.172	.135

TABLE 12

Ipsative Validity Covariances and Coefficients

Variable	Covariance	Coefficient
1	.163	.228
2	−.342	−.369
3	.004	.005
4	−.015	−.024
5	.205	.320
6	−.015	−.023
Σ	.000	.137

If the premultiplication by the diagonal matrix is eliminated from (3), the balance gives the vector of validity covariances for the case when the criterion has a standard deviation of one. Table 12 gives the vectors of validity covariances and coefficients for the six predictor variables. Note that the sum of the validity covariances is zero. This will always be true for ipsative variables. Note also that the validity coefficients tend to sum to zero and are generally less than the original values. For instance, compare the corresponding values in Tables 8 and 12.

In comparing the correlation matrix and vector of validity coefficients for the original variables with the ipsative set, the interesting thing to note is the far greater number of negative correlations for the ipsative set. This phenomenon will be discussed further in Chapter IV.

Relation of Multiple Correlations

The major purpose for assessing human behavior is to predict the outcome, or some aspect of the outcome, of a future event of social significance. Rarely does a single measure supply enough information to do an adequate job of prediction; in most instances researchers are faced with the problem of combining various measures in order to predict the criterion at hand effectively.

Procedures are well known for efficiently combining measures in a linear fashion in order to predict best a criterion in the least-square sense. The correlation between this best combination or weighted sum of measures and the criterion is given by R, the coefficient of multiple correlation. The exact method for computing R involves computing an inverse for the matrix of predictor intercorrelations. However, it is only possible to compute an inverse for matrices belonging to the square basic or nonsingular class, and ipsative intercorrelation matrices do not belong to this class.

Even though it is not possible to compute the inverse for an ipsative intercorrelation matrix, it is still possible to obtain a set of weights which will yield a best least-square estimate of the criterion at hand. Actually, it is possible to utilize several methods to compute the multiple correlation between a criterion and an ipsative set of variables, but the technique most analogous to the inverse approach is the *general inverse* solution developed by Horst [9]. This is an "exact" solution and is the one which will be utilized for comparing the R's obtained from ipsative scores with the R's obtained from raw scores.

The relation between the square of the multiple correlation coefficients for these two types of variables can be expressed as

$$ {}_iR^2 = R^2 - \frac{\beta' 1\, 1' \beta}{1' r^{-1} 1} , \tag{4} $$

where

${}_iR^2$ is the square of the ipsative multiple correlation coefficient,
R^2 is the square of the *raw* score multiple correlation coefficient,
β is the vector of β weights for the raw score case,
r^{-1} is the inverse of the original or *raw* score intercorrelation matrix.

The term subtracted from R^2 in (4) is always positive or zero; hence ${}_iR$ is always equal to or less than R. For the numerical example of this study R was found to be .591 and ${}_iR$ was .470. Thus considerably more information relating to the criterion was available in the raw score matrix than in the ipsative score matrix.

Effect of Deleting a Variable

It should be obvious that in most instances an ipsative matrix can be made basic or nonsingular by the deletion of any single variable. Specifically this applies when the intercorrelation matrix for the variables in raw score form is basic. However, the researcher unfamiliar with matrix theory may find it difficult to accept the idea that any single variable can be removed from an ipsative set without affecting the validity of the battery, especially in view of the variability of the validity coefficients (see e.g., Table 12).

To illustrate this point further, consider the research worker interested in selecting candidates to be trained for the ministry. Let us say that this researcher used as a measuring instrument the Allport-Vernon *Study of Values*, and found that the Religious variable had the highest validity coefficient. He may find it difficult indeed to believe that this variable could be deleted from the set without in the least affecting his ability to predict success in a ministerial training program. Nevertheless, it is true that using only the Theoretical, Economic, Aesthetic, Social, and Political scores the multiple correlation will be identical with that which would have been obtained if the Religious scale had been included. On the other hand, deleting a predictor from the raw score matrix will have a variable effect on the multiple correlation. The resulting R will always be less than the original R

except in the case where the β coefficient was zero for the variable deleted. The actual decrement is given by

$$R_d^2 = R^2 - \frac{\beta_i^2}{(r^{-1})_{ii}}, \tag{5}$$

where

R_d is the multiple correlation after the deletion of variable i,
R is the original multiple correlation,
β_i is the original β coefficient corresponding to variable i,
$(r^{-1})_{ii}$ is the diagonal element from the original inverse corresponding to the variable deleted.

Table 13 shows the decrement in R occasioned by the deletion of different variables. It will be noted that after the deletion of either raw variable 3 or 6 the R did not decrease from the original value. This was so because the β coefficients for these two variables were near zero. In this example, the information in the predictor variables relevant to the criterion is greater after the deletion of any single variable than all the information contained by the ipsative variables. As a matter of fact, the zero-order validity coefficients for raw variables 1 and 5 (.487 and .529) are both greater than the multiple correlation obtained with all of the ipsative variables.

Relation of Ipsative Covariance Matrices to First-Centroid Residual Matrices

It has already been noted that ipsative intercorrelation matrices are nonbasic and hence have a rank of at most one less than their order. The ipsative covariance matrix also has this property, as the rank of a covariance matrix is always the same as the rank of its corresponding correlation matrix.

TABLE 13

Effect on Multiple Correlation of Deleting a Variable

Variable Deleted	Scores in Raw Units	Scores in Ipsative Units
	R	${}_iR$
None	.591	.470
1	.568	.470
2	.579	.470
3	.591	.470
4	.586	.470
5	.529	.470
6	.591	.470

It has also been noted that the sums of the columns (or rows) of an ipsative covariance matrix are all zero. These two properties also hold for any first-centroid residual matrix.

Both the first-centroid residual and the ipsative covariance matrix can be obtained by simple transformations on the intercorrelation matrix of the variables prior to ipsatizing. For easy comparison (1), which gives the transformation yielding the ipsative covariance matrix, is repeated below:

$$C = \left(I - \frac{1\,1'}{m}\right) r \left(I - \frac{1\,1'}{m}\right). \tag{1}$$

The formula for the first-centroid residual (F) can be shown to be

$$F = r - \frac{r1\,1'r}{1'\,r1}\,, \tag{6}$$

where

r is the original intercorrelation matrix,
1 is a unit or summing vector.

The above formulas and discussion may be interpreted as implying that the physical relationship between the ipsative covariance matrix and the first-centroid residual is closer than the physical relationship existing between the ipsative correlation matrix and the first-centroid residual. This certainly is true for the case where the original variables were in normal units with means of zero and standard deviations of unity. It will be proved that under these conditions the ipsative covariance matrix is identically equal to the first-centoid residual when the column (or row) sums of the intercorrelation matrix for the variables in raw form are all equal.

In the empirical example of this study the column sums of the intercorrelations for the pre-ipsative variables are far from equal as shown in Table 8. The smallest of these values is 1.777 and the largest is 3.147. Considering the small number of variables, this is a rather large difference, hence the requirement for algebraic identity is not met. Even so, the similarity between the ipsative covariance matrix and the first-centroid residual is striking. Table 14 gives these two matrices and also the matrix representing their differences. For easy comparison all three matrices were rounded to two places. It is clear from Table 14 that the ipsative covariance matrix and the first-centroid residual based on the same original set of variables are in this example identical for all practical purposes. This is in spite of the fact that the column sums for the original correlation matrix were quite variable, thus not meeting the condition for absolute identity specified earlier.

Number of Negative Values in Ipsative Intercorrelation Matrices

It was mentioned in Chapter I that Guilford [6] has stated that correlation matrices based on ipsative variables have about two-thirds of their

TABLE 14

Comparison of Ipsative Covariance Matirx C and First-Centroid Residual F

Variable	C Matrix					
	1	2	3	4	5	6
1	.51	−.31	.14	−.20	.03	−.18
2	−.31	.86	−.41	.13	−.16	−.11
3	.14	−.41	.69	−.25	−.10	−.08
4	−.20	.13	−.25	.40	−.10	.01
5	.03	−.16	−.10	−.10	.41	−.08
6	−.18	−.11	−.08	.01	−.08	.44
Σ	−.01	.00	−.01	−.01	.00	.00
Variable	F Matrix					
1	.51	−.31	.15	−.20	.03	−.18
2	−.31	.81	−.43	.16	−.14	−.09
3	.15	−.43	.68	−.24	−.09	−.07
4	−.20	.16	−.24	.39	−.11	.00
5	.03	−.14	−.09	−.11	.40	−.09
6	−.18	−.09	−.07	.00	−.09	.43
Σ	.00	.00	.00	.00	.00	.00
Variable	Difference Matrix ($C - F$)					
1	.00	.00	−.01	.00	.00	.00
2	.00	.05	.02	−.03	−.02	−.02
3	−.01	.02	.01	−.01	−.01	−.01
4	.00	−.03	−.01	.01	.01	.01
5	.00	−.02	−.01	.01	.01	.01
6	.00	−.02	−.01	.01	.01	.01
Σ	−.01	.00	−.01	−.01	.00	.00

elements negative. Actually, the number of negative values will depend on two factors: the number of variables and the variability of the intercorrelations.

Under certain assumptions the equation which gives the proportion of negative values $P_{(-)}$ is

$$P_{(-)} = \int_{-\infty}^{1/m\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{7}$$

where σ equals the standard deviation of all of the off-diagonal elements in the original intercorrelation matrix, and m is the number of variables. This equation gives only an approximate estimate of the number of negative values. Nevertheless, even with the additional assumption that the variance

TABLE 15

Actual Number of Negative Values in a Number of Ipsative Intercorrelation Matrices Compared with Number of Negative Values Estimated by Formula (7)

Source	σ	Number of Correlations	$P_{(-)}$	Number of Negative Values Estimated	Number of Negative Values Actual
Thesis Example	.2406	15	.755	11	11
Manual of Allport-Vernon *Study of Values* [2]					
Female Sample	.2360	15	.761	11	11
Male Sample	.2017	15	.796	12	13
Allport *Study of Values* Matrix reported by Ferguson, *et al.* [5]	.2232	15	.773	12	12
Manual of Edwards' PPS [4]	.1724	105	.651	68	68
Manual of Kuder Pref. Record, Form C (12)	.2120	45	.681	31	33

of the off-diagonal elements in an ipsative correlation matrix is equivalent to the variance of the corresponding elements in the original intercorrelation matrix, this formula can be used to estimate quite accurately the number of negative elements in an ipsative intercorrelation matrix. This is illustrated in Table 15.

Actually all that is necessary to estimate the number of negative values is to determine the standard deviation of the off-diagonal values, then compute the reciprocal of the product of that value by the number of variables, and finally, using this reciprocal as the abscissa, determine $P_{(-)}$ from a normal curve table. Of course, the number of negative values could be determined much more easily by merely counting them. However, the close correspondence between the estimated and actual number of negative correlations in Table 15 serves to emphasize the dependence of the signs of the intercorrelations in an ipsative set on (i) the variability of the correlations and (ii) the number of variables in the set. Furthermore, if we accept the underlying assumptions (see Chapter IV), then it is clear from (7) that the proportion of negative ipsative intercorrelations is always 50 per cent or greater inasmuch as the upper limit of the integral can never be less than zero. It is also clear from (7) and from Table 15 that the proportion of negative values tends to increase as the number of variables decreases. It is interesting to note that in only one instance does the estimated number of negative values differ from the actual number of negative values by more than one. In that one case the difference was only two, and the test was the Kuder which is the only test in Table 15 which is not perfectly ipsative.

CHAPTER IV

MATHEMATICAL DERIVATIONS

In the preceding chapters several statements were made without proof concerning the properties of ipsative variables. In this chapter proofs will be presented for those statements and for other propositions not previously stated. All developments will be given in matrix algebra notation.

1. *Ipsative Intercorrelation and Covariance Matrices*

Property 1.1

An ipsative intercorrelation matrix can be expressed as a simple function of the matrix of intercorrelations for the same variables prior to ipsatizing.

Definitions:

N is the number of cases. (For simplicity in notation N is considered of sufficient size to avoid the necessity of considering degrees of freedom.)
m is the number of variables.
X is the matrix of standard scores, each variable having a mean of zero and a standard deviation of one.
Y is the matrix of ipsative scores corresponding to X.
r is the intercorrelation matrix of the X variables.
1 is a unit or summing vector.

By the definition of an ipsative matrix given earlier it is known that

$$Y1 = K1, \tag{8}$$

where K is a constant. Without loss of generality, as far as the intercorrelations of the variables in Y are concerned, K can be assumed to equal zero. Then the transformation on X which yields Y is

$$Y = X\left(I - \frac{1\,1'}{m}\right). \tag{9}$$

From (9) it is clear that Y is obtained by transforming the elements of X into deviation scores by rows. It is also clear from (9) and from the definition of X that the variables in Y are in deviation scores by columns. Hence

$$1'Y = 0'. \tag{10}$$

Now if we let C be the covariance matrix for the Y variables then

$$C = \frac{Y'Y}{N}. \tag{11}$$

Substituting (9) in (11)

$$C = \left(I - \frac{1\,1'}{m}\right)\frac{X'X}{N}\left(I - \frac{1\,1'}{m}\right). \tag{12}$$

From the definitions it is clear that

$$r = \frac{X'X}{N}\,; \tag{13}$$

hence substituting (13) in (12) we have

$$C = \left(I - \frac{1\,1'}{m}\right)r\left(I - \frac{1\,1'}{m}\right). \tag{14}$$

The diagonal elements of C are of course the variances of the corresponding variables from Y. Letting D_C be the diagonal matrix corresponding to these variances we may write the ipsative intercorrelation matrix $({}_ir)$ as

$${}_ir = D_C^{-1/2}\left(I - \frac{1\,1'}{m}\right)r\left(I - \frac{1\,1'}{m}\right)D_C^{-1/2}. \tag{15}$$

Equations (14) and (15) make it clear that, when the intercorrelation matrix is available for the variables prior to ipsatizing, the ipsative intercorrelation matrix is readily obtained by "double centering" the r matrix and then pre- and postmultiplying the resultant matrix by the reciprocals of the square roots of its diagonal entries.

Property 1.2

The sums of the columns (or rows) of an ipsative covariance matrix must always equal zero.

Starting directly with the variables in ipsative form it is first necessary to convert them into deviation scores by columns. This is accomplished by premultiplying the Y matrix by a centering matrix as in (16).

$$y = \left(I - \frac{1\,1'}{N}\right)Y. \tag{16}$$

The covariance matrix is then given by

$$C = \frac{y'y}{N}. \tag{17}$$

Substituting (16) in (17)

$$C = \frac{Y'\left(I - \frac{1\,1'}{N}\right)\left(I - \frac{1\,1'}{N}\right)Y}{N}. \tag{18}$$

As the matrices in parentheses are idempotent we can write

$$C = \frac{Y'\left(I - \frac{1\,1'}{N}\right)Y}{N}. \tag{19}$$

Summing by columns and multiplying out we have

$$1'C = \frac{1'Y'Y - \frac{1'Y'1\,1'Y}{N}}{N}. \tag{20}$$

Substituting (8) in (20),

$$1'C = \frac{K1'Y - \frac{K1'1\,1'Y}{N}}{N}, \tag{21}$$

and (20) can be further simplified as

$$1'1 = N;$$

thus

$$1'C = \frac{K1'Y - K1'Y}{N}, \tag{22}$$

or

$$1'C = 0'. \tag{23}$$

Q.E.D.

Property 1.3

In the special case where the ipsative variances are equal, the sums of the columns (or rows) of the ipsative intercorrelation matrix are equal to zero.

Under the restriction stated in the above property, (15) sums to

$$1'({}_ir) = a^2(1' - 1')r\left(I - \frac{1\,1'}{m}\right), \tag{25}$$

where a is a constant, and this expression reduces to

$$1'({}_ir) = 0' \tag{26}$$

Q.E.D.

Property 1.4

Ipsative intercorrelation matrices are nonbasic or singular and thus have no regular inverse.

First we recall (15)

$$(15)\qquad {}_ir = D_C^{-1/2}\left(I - \frac{1\,1'}{m}\right)r\left(I - \frac{1\,1'}{m}\right)D_C^{-1/2}.$$

The matrices in parentheses are special cases of Guttman's [7] rank-reduction theorem and therefore have rank one less than their order. Their order is of course the same as the order of r. It is well known that the maximum rank of a matrix resulting from a series of matrix multiplications cannot exceed the rank of the factor of lowest rank. Therefore, it immediately follows that ${}_ir$ is not basic.

2. *Validity Coefficients for an Ipsative Set*

Property 2.1

The validity coefficients for an ipsative set of variables can be expressed as a function of the intercorrelation matrix and vector of validity coefficients for the same variables prior to ipsatizing.

Additional definitions:

- V is a vector of criterion scores with mean of zero and variance of unity.
- r_c is a vector of validity coefficients for the variables in pre-ipsative form.
- ${}_ir_c$ is the vector of validity coefficients for the ipsatized variables.
- D_C is the diagonal matrix of the variances of the ipsatized variables.

Because of (10) and the above definitions we can write

$$(27)\qquad {}_ir_c = D_C^{-1/2}\frac{Y'V}{N};$$

from (9) in (27)

$$(28)\qquad {}_ir_c = D_C^{-1/2}\frac{\left(I - \frac{1\,1'}{m}\right)X'V}{N}.$$

By definition

$$(29)\qquad r_c = \frac{X'V}{N}.$$

Substituting (29) in (28) we have

$$(30) \qquad {}_ir_c = D_C^{-1/2}\left(I - \frac{1\,1'}{m}\right)r_c.$$

From (14) and (30) it is clear that the vector of ipsative validity coefficients can readily be obtained as a function of the intercorrelation matrix and vector of validity coefficients for the same variables prior to ipsatizing.

Property 2.2

The sum of the covariance terms obtained between a specified criterion and a set of ipsative variables is zero.

Let T be the vector of covariance terms between the specified criterion and the set of ipsative variables.

$$(31) \qquad T = \frac{Y'V}{N};$$

from (9) in (31)

$$(32) \qquad T = \frac{\left(I - \frac{1\,1'}{m}\right)X'V}{N}$$

Summing the elements of T we have

$$(33) \qquad 1'T = \frac{1'\left(I - \frac{1\,1'}{m}\right)X'V}{N}$$

which simplifies to

$$(34) \qquad 1'T = \frac{(1' - 1')X'V}{N}$$

or

$$(35) \qquad 1'T = 0'. \qquad \text{Q.E.D.}$$

Property 2.3

In the special case where the ipsative variances are all equal the sum of the ipsative validity coefficients is zero.

In the special case stated in Property 2.3, (30) can be rewritten as

$$(36) \qquad {}_ir_c = KI\left(I - \frac{1\,1'}{m}\right)r_c,$$

where K is the reciprocal of the constant standard deviation. Summing the validity coefficients in (36) we have

$$(37) \qquad 1'({}_ir_c) = K(1' - 1')r_c$$

or

$$1'({}_ir_c) = 0'. \tag{38}$$ Q.E.D.

3. *Multiple Correlation in Ipsative Case*

Property 3.1

The multiple correlation of a set of ipsative variables with a criterion can be expressed as a function of the multiple correlation for the pre-ipsatized variables and their beta-coefficients, and except in one rare special case the ipsative multiple correlation is always less than the multiple correlation for the same variables prior to ipsatizing.

It is well known that in most instances the square of the multiple correlation of a set of variables with a criterion is given by the formula

$$R^2 = r_c' r^{-1} r_c, \tag{39}$$

or

$$R^2 = \frac{1}{N} V'X(X'X)^{-1}X'V, \tag{40}$$

where R is the multiple correlation and the other symbols are as previously defined. But these formulas cannot be applied to the ipsative case because the ipsative intercorrelation matrix has been proved to be nonbasic or singular and thus has no regular inverse. However, it was mentioned in Chapter III that an analagous procedure, yielding an exact least-square solution for the regression weights, has been developed by Horst [9]. This procedure yields a "general" inverse which can be manipulated in much the same manner as the regular inverse. If we let parentheses around the regular inverse symbol, i.e., (-1), symbolize the general inverse in (41), the multiple-correlation equation for an ipsative set of variables with a criterion which is analagous to (40) is from (9) and (12)

$$ {}_iR^2 = \frac{1}{N} V'X\left(I - \frac{1\,1'}{m}\right)\left[\left(I - \frac{1\,1'}{m}\right)(X'X)\left(I - \frac{1\,1'}{m}\right)\right]^{(-1)} \cdot\left(I - \frac{1\,1'}{m}\right)X'V. \tag{41}$$

If we let

$$X'X = S, \tag{42}$$

then it can be readily proved that the general inverse, W, of the matrix in the brackets in (41) is given by

$$W = \left(S^{-1} - \frac{S^{-1}1\,1'S^{-1}}{1'S^{-1}1}\right). \tag{43}$$

Substituting (43) in (41)

$$(44) \qquad {}_iR^2 = \frac{1}{N} V'X\left(I - \frac{1\,1'}{m}\right)\left(S^{-1} - \frac{S^{-1}1\,1'S^{-1}}{1'S^{-1}1}\right)\left(I - \frac{1\,1'}{m}\right)X'V,$$

which immediately simplifies further, because in this instance the unit vector is orthogonal to the general inverse, to

$$(45) \qquad {}_iR^2 = \frac{1}{N} V'X\left(S^{-1} - \frac{S^{-1}1\,1'S^{-1}}{1'S^{-1}1}\right)X'V.$$

Because of (13) and (42) we note that

$$(46) \qquad S^{-1} = \frac{r^{-1}}{N}.$$

From (13), (29), (42), and (46) in (45) we have

$$(47) \qquad {}_iR^2 = r_c'r^{-1}r_c - \frac{r_c'r^{-1}1\,1'r^{-1}r_c}{1'r^{-1}1}.$$

It is well known that

$$(48) \qquad \beta' = r_c'r^{-1},$$

and from (39) and (48) in (47) we have

$$(49) \qquad {}_iR^2 = R^2 - \frac{\beta'1\,1'\beta}{1'r^{-1}1}. \qquad \text{Q.E.D.}$$

Equation (49) makes it clear that the square of the multiple correlation for the ipsative set is equivalent to the square of the multiple correlation obtained with the pre-ipsative variables less the square of the sum of the β coefficients divided by the sum of all the elements in the inverse of the pre-ipsatized intercorrelation matrix.

If the fraction on the right in (49) is positive, ${}_iR$ will be less than R. The numerator term can never be negative as it is the square of the sum of the β coefficients. The denominator can also never be negative as it is the sum of all the elements in a Gramian matrix. Therefore, except in the rare case where the sum of the β coefficients for the pre-ipsatized variables is exactly zero, ${}_iR$ will always be less than R. When the sum of the β coefficients is equal to zero ${}_iR$ will equal R.

4. *Effect on R of Deleting a Variable from Ipsative Set*

Property 4.1

The least-square estimate of a criterion using all of the variables of an ipsative set is identical with the least-square solution with any single variable deleted.

As in the above developments we let X be an $N \times m$ basic matrix, and, rewriting (9),

$$Y = X\left(I - \frac{1\,1'}{m}\right); \tag{9}$$

we note that the rank of Y is $m - 1$. Hence we can let

$$Y = uv', \tag{50}$$

where the common order of u and v' is $m - 1$.

Now consider the expression

$$YB - Z = E, \tag{51}$$

where Z is a vector which we wish to estimate from the matrix Y, and B is a vector of least-square regression weights.

Substituting (50) in (51) we have

$$uv'B - Z = E. \tag{52}$$

Let

$$v'B = b. \tag{53}$$

Then from (53) in (52), we have,

$$ub - Z = E. \tag{54}$$

It is well known that the solution for b that best satisfies (54) in the least-square sense is

$$b = (u'u)^{-1}u'Z. \tag{55}$$

Referring to (53), we note that for the solution of B there are more unknowns than equations; hence an infinite number of solutions exist. It can be easily demonstrated that one such solution is

$$B = v(v'v)^{-1}b. \tag{56}$$

This is the general inverse solution developed by Horst, and it has also been proved [9] to be a least-square solution for the weights. Although other solutions for the weights exist they will have a larger sum of squares. Inasmuch as frequently some of the weights are negative, solutions other than the general inverse will tend to have larger negative weights. Experience has shown that this will lead to unstable estimates. Thus the general inverse solution, though only one of an infinite number, is generally to be preferred.

From (55) in (56)

$$B = v(v'v)^{-1}(u'u)^{-1}u'Z. \tag{57}$$

Substituting (57) in (52) we have

(58) $$uv'v(v'v)^{-1}(u'u)^{-1}u'Z - Z = E,$$

which reduces to

(59) $$u(u'u)^{-1}u'Z - Z = E.$$

The vector indicated in (59) by $u(u'u)^{-1}u'Z$ is the best least-square estimate of Z using all the variables in Y. The next steps in the development will prove that the least-square solution given by (59) is identically equivalent with the least-square solution after the deletion of any single variable from Y (indicated by $Y_{(i)}$).

From (50) it is obvious that v' has as many columns as Y, and if we delete a column from Y the equality indicated in (50) can be maintained by deleting the corresponding column from v'. This is indicated below as

(60) $$Y_{(i)} = uv'_{(i)},$$

where the i's in parentheses indicate that the ith column has been deleted from both Y and v'.

The best estimate of Z will now depend on the determination of $B_{(i)}$ in the expression

(61) $$Y_{(i)}B_{(i)} - Z = E.$$

Substituting from (60) in (61)

(62) $$uv'_{(i)}B_{(i)} - Z = E,$$

and letting

(63) $$v'_{(i)}B_{(i)} = f,$$

we can rewrite (62) as

(64) $$uf - Z = E,$$

and analogous to (55) and (57) we have

(65) $$f = (u'u)^{-1}u'Z,$$

and

(66) $$B_{(i)} = v_{(i)}(v'_{(i)}v_{(i)})^{-1}(u'u)^{-1}u'Z.$$

Substituting (66) in (62)

(67) $$uv'_{(i)}v_{(i)}(v'_{(i)}v_{(i)})^{-1}(u'u)^{-1}u'Z - Z = E,$$

which reduces to

(68) $$u(u'u)^{-1}u'Z - Z = E.$$

Equation (68) proves the identity of the least-square estimate of Z from $Y_{(i)}$ with that of Z from Y as it is identical with (59).

Lest the reader unfamiliar with matrix algebra should get the impression that the same solution would be found if two or more columns were deleted from Y, attention should be called to (66). It is clear from (66) that a solution cannot be found for $B_{(i)}$ unless an inverse exists for $v'_{(i)}v_{(i)}$ but, because the original order of v' was $m - 1$ by m, if more than one column is deleted then $v'_{(ij)}v_{(ij)}$ will be a major rather than a minor product and, of course, an inverse will not exist.

5. *Effect of Deleting Variable from Raw Score Set*

In the previous section it was proved that the deletion of a variable from an ipsative set did not decrease the information in the set. (Here the word "information" is used synonymously with the phrase "ability to predict a criterion.") On the other hand, the deletion of a variable from a raw score set will in general decrease the information available. Of course, the decrement in this case will depend on the variable deleted as illustrated in Table 13 of Chapter III.

Although no decrement in information is occasioned by the deletion of a single variable from an ipsative set, there is in general a decrement in information caused by the transformation from raw to ipsative units (see Property 3.1). It is interesting to compare this information loss with that occasioned by the deletion of a variable from a raw score set. This has been done empirically in Table 13 of Chapter III. In this section the equation will be developed for the loss in information occasioned by the deletion of a variable from a raw set in order that it can be compared with the expression for the information loss brought about by the conversion from raw to ipsative units.

Property 5.1

The loss of ability to predict a criterion occasioned by the deletion of a variable from a raw set can be expressed as a function of the corresponding β coefficient and the corresponding diagonal element of the original inverse. (This is not an ipsative property, but it is necessarily included for comparison with the ipsative case.)

First we let r be the matrix of intercorrelations of the variables in pre-ipsative form. Next, without loss of generality, we can assume that the variable which we wish to delete is represented in r by the last or mth row and column. We will let r_d be the matrix of intercorrelations of the remaining variables. Then as Horst [9] has shown

$$\left[\begin{array}{c:c} r_d^{-1} & 0 \\ \hdashline 0' & 0 \end{array}\right] = r^{-1} - \frac{(r^{-1})_{.m}(r^{-1})'_{m.}}{(r^{-1})_{mm}}, \tag{69}$$

where

- $0'$ is a null row-vector,
- $(r^{-1})_{.m}$ is the last column of r^{-1} including the diagonal element,
- $(r^{-1})'_{m.}$ is the last row of r^{-1} including the diagonal element,
- $(r^{-1})_{mm}$ is the diagonal element.

Letting r_c be the vector of validity coefficients, we have

$$r_c = \begin{bmatrix} r_{cd} \\ \cdots \\ r_{cm} \end{bmatrix}, \tag{70}$$

where

- r_{cd} is the vector of validity coefficients corresponding to the variables in r_d,
- r_{cm} is the validity coefficient corresponding to the variable which is to be deleted.

In the foregoing the subscript d was used to designate the intercorrelation matrix and the vector of validity coefficients which remained after the variable in question had been deleted. Consistent with that notation we will let R_d be the multiple correlation between the remaining predictor variables and the criterion. Then, analogous to (39), we may write

$$R_d^2 = r'_{cd} r_d^{-1} r_{cd}. \tag{71}$$

Using the partitioned matrices given in(69) and (70), we can rewrite

$$r'_{cd} r_d^{-1} r_{cd} = [r'_{cd} \vdots r_{cm}] \begin{bmatrix} r_d^{-1} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0' & \vdots & 0 \end{bmatrix} \begin{bmatrix} r_{cd} \\ \cdots \\ r_{cm} \end{bmatrix}. \tag{72}$$

From (69), (70), and (71) in (72)

$$R_d^2 = r'_c\left(r^{-1} - \frac{(r^{-1})_{.m}(r^{-1})'_{m.}}{(r^{-1})_{mm}}\right)r_c, \tag{73}$$

which multiplies out to

$$R_d^2 = r'_c r^{-1} r_c - \frac{r'_c(r^{-1})_{.m}(r^{-1})'_{m.}r_c}{(r^{-1})_{mm}}. \tag{74}$$

It is well known that β, the vector of beta coefficients, is given by

$$\beta = r^{-1} r_c, \tag{75}$$

and as $(r^{-1})'_{m.}$ is simply the mth row out of r^{-1} it immediately follows that

$$\beta_m = (r^{-1})'_{m.} r_c. \tag{76}$$

Hence we observe β_m is the mth element from the vector of β coefficients.

Of course where R is the multiple correlation of the predictors with the criterion when no variables have been deleted we can write

$$(77) \qquad R^2 = r_c' r^{-1} r_c.$$

Substituting (76) and (77) in (74)

$$(78) \qquad R_d^2 = R^2 - \frac{\beta_m^2}{(r^{-1})_{mm}}.$$

The ratio to the right of the negative sign in (78) indicates the decrement in the square of the multiple correlation brought about by the deletion of the mth variable. It should be noted that both terms in this ratio are always positive—the numerator is squared and the denominator is the diagonal element of the inverse of a Gramian matrix. Inasmuch as neither the beta coefficients nor the elements of the inverse are altered in any way except location by rearranging the order of the variables, (78) can be rewritten for the general case as

$$(79) \qquad R_d^2 = \underline{R}^2 - \frac{\beta_i^2}{(r^{-1})_{ii}},$$

where

β_i is the β coefficient corresponding to the ith variable,
$(r^{-1})_{ii}$ is the ith diagonal element from the inverse of full order.

6. *Negative Values in Ipsative Intercorrelation Matrix*

It has already been proved (Property 1.2) that the column sums of an ipsative covariance matrix must be zero. It was also proved (Property 1.3) that when the ipsative variances are equal the column sums of the ipsative intercorrelations are zero. Both of these properties make it clear that ipsative intercorrelation matrices will have a large proportion of negative values. It will be shown in this section that the actual proportion of negative values varies as a function of (i) the number of variables and (ii) the variance of the original intercorrelations. This has already been demonstrated empirically in Table 15 of Chapter III.

Unless some assumptions are made about the relationships of the preipsatized variables, it is very difficult to determine an expression which will give the proportion of negative values in the resulting ipsative intercorrelation matrix. For this reason three developments will be presented. First, the assumption will be made that the original variables are orthogonal. Second, the assumption will be made that all of the original variables are correlated with each other to the same degeee. And, third, the less restrictive case will be considered where the off-diagonal elements are not necessarily constant, but the column sums of the original intercorrelation matrix are all equal.

Property 6.1

When the variables in non-ipsative form are orthogonal, the ipsative intercorrelations will all be a negative constant value determined only by the number of variables.

Under the restriction stated in Property 6.1, r, as defined in (13), becomes the identity matrix, and the covariance matrix, C, given in (14) simplifies to

$$C = I - \frac{1\,1'}{m}. \tag{80}$$

It is clear that the diagonal elements of (80) are given by

$$D_C = \left(\frac{m-1}{m}\right)I, \tag{81}$$

$$D_C^{-1/2} = \sqrt{\frac{m}{m-1}}\,I, \tag{82}$$

and from (15), (80), and (82) we have

$${}_ir = \frac{m}{m-1}\left(I - \frac{1\,1'}{m}\right). \tag{83}$$

From (83) it is clear that under the assumption that the original variables were orthogonal, the resulting ipsative matrix of intercorrelations has the property that all of its off-diagonal elements are negative and constant in value, the typical off-diagonal element being

$${}_ir_{ij} = -\frac{1}{m-1}. \tag{84}$$

Property 6.2

When the variables in non-ipsative form are all correlated with each other to some constant degree, a, the ipsative intercorrelations will be exactly the same as if the original variables had been orthogonal as in Property 6.1.

Under the restriction given we have

$$r = I + a(1\,1' - I), \tag{85}$$

and substituting (85) in (14), we have

$$C = \left(I - \frac{1\,1'}{m}\right)[I + a(1\,1' - I)]\left(I - \frac{1\,1'}{m}\right), \tag{86}$$

which simplifies to

$$C = (1 - a)\left(I - \frac{1\,1'}{m}\right). \tag{87}$$

From (87), it follows that

$$D_C = \frac{(1 - a)(m - 1)}{m} I, \tag{88}$$

and

$$D_C^{-1/2} = \sqrt{\frac{m}{(1 - a)(m - 1)}}\, I. \tag{89}$$

From (14), (15), (87), and (89) we have

$${}_ir = \frac{m}{(1 - a)(m - 1)} (1 - a)\left(I - \frac{1\,1'}{m}\right), \tag{90}$$

which reduces to

$${}_ir = \frac{m}{m - 1}\left(I - \frac{1\,1'}{m}\right). \tag{91}$$

As (91) is identically equal to (83), Property 6.2 is proved.

Property 6.3

When the column sums of the pre-ipsative intercorrelation matrix are all equal and the off-diagonal correlations are distributed normally, the proportion of negative values in the ipsative intercorrelation matrix can be determined readily from tabled values of the normal curve.

Under the assumption that the sum of the columns (or rows) of the original intercorrelation matrix are all equal we have

$$r1 = k1, \tag{92}$$

where k is the sum of any column. Expanding (14), we have

$$C = r - \frac{1\,1'r}{m} - \frac{r1\,1'}{m} + \frac{1\,1'r1\,1'}{m^2}. \tag{93}$$

Substituting (92) in (93) and simplifying we have

$$C = r - \frac{k1\,1'}{m}. \tag{94}$$

The typical element in C is given by

$$C_{ij} = r_{ij} - \frac{k}{m}. \tag{95}$$

Because of (92), k/m is equal to the mean of all of the elements in r including the diagonal elements of unity. The mean of the off-diagonal elements will, of course, always be less than k/m. Thus, under the assumption that the off-diagonal elements are distributed normally, the number of negative off-diagonal elements in the ipsative covariance or correlation matrix will always exceed fifty per cent.

It should also be clear that the greater the number of variables the closer k/m will approximate the mean of the off-diagonal elements. This is true because the effect of the unit elements in the diagonal is less as the total number of elements increases. Hence, the number of negative intercorrelations is a function of m, the number of variables.

Let Σ be the sum of all but the diagonal element in any column; then

$$\frac{k}{m} = \frac{\Sigma}{m} + \frac{1}{m}. \tag{96}$$

Now, under the assumption that the off-diagonal intercorrelations are distributed normally, the subtraction of Σ/m from these elements would make precisely one half of them negative. It is clear the subtraction of the *additional amount*, $1/m$, as indicated in (95), will make more than one half of the off-diagonal elements negative. Under the assumptions given in Property 6.3, the proportion of negative values in the off-diagonal elements will be given by

$$P_{(-)} = \int_{-\infty}^{1/m\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{97}$$

where σ in (97) is the standard deviation of the off-diagonal elements in the original intercorrelation matrix, and m is the number of variables.

Formula (97) holds because the "zero" point in the normal distribution has been shifted $1/m$ units in the positive direction for the reasons discussed above. The shift in standard units is $1/m\sigma$. The area under the normal curve from minus infinity to $1/m\sigma$ gives the proportion of negative values which will be found in the ipsative intercorrelation matrix.

7. *Relation of Ipsative Covariance Matrices to First-Centroid Residual*

In Table 14 of Chapter III an empirical example was presented showing the close resemblance between the first-centroid residual of an intercorrelation matrix and the ipsative covariance matrix obtained from the same original variables in normalized form (mean of zero and standard deviation of unity). In this section a proof will be presented showing that these two matrices are identical under the restriction that the sums of the columns (or rows) of the original intercorrelation matrix are all equal.

Property 7.1

When the original scores are in normalized units with a mean of zero and standard deviation of unity and the sums of the columns (or rows) of the original intercorrelation matrix are all equal, then the ipsative covariance matrix and the first-centroid residual are identical.

Because of the restriction on r, (92) still holds

$$r1 = k1. \tag{92}$$

Letting F be the first-centroid residual we have

$$F = r - \frac{r1\ 1'r}{1'r1}, \tag{98}$$

and substituting (92) in (98)

$$F = r - \frac{k1\ 1'}{m}, \tag{99}$$

which is identically equivalent to the ipsative covariance matrix as given in (94). Q.E.D.

Subtracting the first centroid from a matrix of course has the effect of removing "information" from the matrix. As a matter of fact, the first centroid approximates Hotelling's [11] first principal component, and, as a consequence, its removal approximates the removal of the greatest portion of variance possible by subtracting from the original matrix the major product of two vectors.

Property 7.2

Under certain circumstances it can be shown that the information loss occasioned by removing the first centroid is equivalent to the information lost by transforming the original scores to ipsative units.

Without loss of generality we can assume that the original scores, X, are in normalized form with mean of zero and standard deviation of one and the criterion scores, V, are also in normalized form. Then the square of the multiple correlation can be written as,

$$R^2 = \frac{1}{N} V'X(X'X)^{-1}X'V, \tag{100}$$

and, when $X'X$ does not have a regular inverse, it can be proved that the square of the multiple correlation can be written as

$$R^2 = \frac{1}{N} V'XWX'V, \tag{101}$$

where W is the general inverse of $X'X$. Under the present score assumptions (98) may be rewritten as

$$F = \frac{1}{N}\left(X'X - \frac{X'X1\ 1'X'X}{1'X'X1}\right), \tag{102}$$

since

$$\frac{X'X}{N} = r, \tag{103}$$

or

$$F = \frac{1}{N} X'\left(I - \frac{X1\,1'X'}{1'X'X1}\right)X. \tag{104}$$

Now let

$$Z = \left(I - \frac{X1\,1'X'}{1'X'X1}\right)X; \tag{105}$$

then as the matrix in parentheses is idempotent

$$Z'Z = X'\left(I - \frac{X1\,1'X'}{1'X'X1}\right)X. \tag{106}$$

From (103), (104), and (105), it is clear that Z can be thought of as a score matrix containing all of the information remaining in the first-centroid residual. Summing the columns of Z as given in (105) we have

$$1'Z = 0'. \tag{107}$$

The matrix Z is thus composed of a set of deviation scores, but, of course, it is not a basic matrix. Indicating the general inverse of $Z'Z$ by

$$W = (Z'Z)^{(-1)}, \tag{108}$$

we can, because of (101), write the square of the multiple correlation of a set of "first-centroid residual variables" as

$${}_cR^2 = \frac{1}{N} V'ZWZ'V. \tag{109}$$

It can be readily proved that the general inverse, W, of $Z'Z$ is

$$W = \left(I - \frac{1\,1'}{1'\,1}\right)(X'X)^{-1}\left(I - \frac{1\,1'}{1'\,1}\right). \tag{110}$$

Substituting from (105) and (110) in (109),

$$\begin{aligned}{}_cR^2 = \frac{1}{N} V\left(I - \frac{X1\,1'X'}{1'X'X1}\right)X\left(I - \frac{1\,1'}{1'\,1}\right)\\ \cdot(X'X)^{-1}\left(I - \frac{1\,1'}{1'\,1}\right)X'\left(I - \frac{X1\,1'X'}{1'X'X1}\right)V,\end{aligned} \tag{111}$$

which simplifies to

$${}_cR^2 = \frac{1}{N} V'X(X'X)^{-1}X'V - \frac{1}{N}\frac{V'X1\,1'X'V}{1'X'X1}. \tag{112}$$

From the assumptions it follows that the validity coefficients of X with V can be expressed as

(113) $$r_c = \frac{1}{N} X'V.$$

From (100), (103), and (113) in (111)

(114) $${}_cR^2 = R^2 - \frac{r_c' 1\ 1' r_c}{1' r 1}.$$

Now rewriting (49), the equation for the square of the multiple correlation obtainable after X has been ipsatized, which is

(49) $${}_iR^2 = R^2 - \frac{\beta' 1\ 1' \beta}{1' r^{-1}\ 1},$$

we observe that the information loss expressed by (114) is identical with the loss in (49) under the assumption that the variables in X were orthogonal, because, under that assumption, β equals r_c and r equals r^{-1}. More generally, the information lost by removing the first centroid is equivalent to the information lost by ipsatizing whenever the two ratios on the right of (114) and (49) are identical.

It is interesting to note that if the original validity coefficients sum to zero no information relevant to the criterion is lost by removing the first centroid. On the other hand, if the original β coefficients sum to zero no information relevant to the criterion is lost by converting the original scores to ipsative units.

8. *Correlation Between Set of Absolute Measures and Their Ipsative Counterparts*

Property 8.1

Under a very special restriction a set of ipsative variables will correlate perfectly with their absolute counterparts.

We will assume that X is the same as defined just previous to (8) and Y is as given in (9). Then the matrix of intercorrelations of the variables in X with their ipsative counterparts can be written as

(115) $$r_{xy} = \frac{1}{N} X'X\left(I - \frac{1\ 1'}{m}\right)D_C^{-1/2},$$

where D_C is as used in (15). Because of the definition of X, (115) can be rewritten

(116) $$r_{xy} = r\left(I - \frac{1\ 1'}{m}\right)D_C^{-1/2}.$$

Equation (116) gives the intercorrelation matrix of a set of absolute measures with their ipsative counterparts.

Under the assumption that the absolute measures were all correlated with each other to some constant degree a we can write

$$r = I + a(1\,1' - I). \tag{117}$$

In order to determine $D_C^{-\frac{1}{2}}$ under this condition we substitute (117) in (14)

$$C = \left(I - \frac{1\,1'}{m}\right)I + a(1\,1' - I)\left(I - \frac{1\,1'}{m}\right), \tag{118}$$

which simplifies to

$$C = (1 - a)\left(I - \frac{1\,1'}{m}\right). \tag{119}$$

From (119) it is clear that

$$D_C^{-1/2} = \sqrt{\frac{m}{(1 - a)(m - 1)}}\, I. \tag{120}$$

Substituting (117) and (120) in (116)

$$r_{xy} = \sqrt{\frac{m}{(1 - a)(m - 1)}}\, [I + a(1\,1' - I)]\left(I - \frac{1\,1'}{m}\right), \tag{121}$$

which simplifies to

$$r_{xy} = \sqrt{\frac{m}{(1 - a)(m - 1)}}\, (1 - a)\left(I - \frac{1\,1'}{m}\right), \tag{122}$$

or

$$r_{xy} = \sqrt{\frac{m(1 - a)}{m - 1}}\left(I - \frac{1\,1'}{m}\right). \tag{123}$$

Actually we are most interested in the diagonal elements of r_{xy} as they are the correlations between each absolute measure and its ipsative counterpart. These diagonal elements are all equal and from (123) they are given by

$$r_{x_iy_i} = \sqrt{\frac{m(1 - a)}{m - 1}}\,\frac{m - 1}{m}, \tag{124}$$

or

$$r_{x_iy_i} = \sqrt{\frac{(m - 1)(1 - a)}{m}}. \tag{125}$$

It can be shown that the constant a in (125) can vary from 1 to $-1/(m - 1)$. With this range of values for a, it may be seen from (125) that in general, a set of ipsative variables, derived from a set of absolute measures correlated with each other to a constant degree, will not correlate perfectly with their absolute counterparts. However, when a has the value $-1/(m - 1)$,

$$r_{x_i y_i} = 1; \tag{126}$$

hence under this very special restriction a set of ipsative variables can correlate perfectly with their absolute counterparts.

9. *Importance of Equating Means and Variances Prior to Ipsatizing*

Definitions:

- N is the number of entities.
- G is an $N \times m$ matrix of raw scores.
- H is the $N \times m$ matrix corresponding to G with equated variances.
- X is the $N \times m$ matrix corresponding to G with equated variances and means of zero.
- D_σ is the diagonal matrix of corresponding standard deviations of G.
- Y is the ipsative matrix obtained by transforming to deviation units the rows of X.
- Z is the ipsative matrix obtained by adding the same constant, k, to each of the elements of Y such that the sum of an entities' scores is some specified value, mk.

From the definitions,

$$H = GD_\sigma^{-1}, \tag{127}$$

$$X = \left(I - \frac{1\ 1'}{N}\right)GD_\sigma^{-1}, \tag{128}$$

$$Y = X\left(I - \frac{1\ 1'}{m}\right), \tag{129}$$

and

$$Z = Y + k1\ 1'. \tag{130}$$

Property 9.1

If the column means of the pre-ipsative matrix are equal, the column means of the resulting ipsative matrix must be equal.

Without loss of generality the means prior ot ipsatizing can be assumed to be zero as in X. The matrix Z is quite general and corresponds to any ipsative matrix obtained from X. The means of Z are from (130)

$$\frac{1'Z}{N} = \frac{1'(Y + k1\ 1')}{N}. \tag{131}$$

Substituting from (129) in (131)

$$\frac{1'Z}{N} = \frac{1'X\left(I - \frac{1\,1'}{m}\right) + kN1'}{N}, \tag{132}$$

but from (128) it is clear that

$$1'X = 0';$$

thus it follows that (132) reduces to

$$\frac{1'Z}{N} = k1', \tag{133}$$

or the means of the variables in Z are all identical and equal to the constant added to the elements of Y.

Property 9.2

If the variances but not the means were equated prior to ipsatizing, transforming the resulting ipsative scores to deviation units will yield interpretable results.

Because of Property 9.1, it can readily be determined whether the means were equated prior to ipsatizing. If the means are not equal, of course, comparison of one score with another for a given entity should not be made. Property 9.2 implies the ipsatizing of a matrix such as H. Hence, we begin by letting T be the ipsative matrix obtained from H or

$$T = H\left(I - \frac{1\,1'}{m}\right) + k1\,1'. \tag{134}$$

Without loss of generality k can be assumed to be zero in which case the rows of T sum to zero, that is

$$T1' = 0, \tag{135}$$

but the sums of the columns of T are quite obviously from (134) a function of the sums of the columns of H and hence are not equal.

Transforming the columns of T to deviation units we have

$$\left(I - \frac{1\,1'}{N}\right)T = \left(I - \frac{1\,1'}{N}\right)H\left(I - \frac{1\,1'}{m}\right). \tag{136}$$

Note that the term $k11'$ from (134) dropped out of (136) as k was assumed to be zero. From (127) and (128) in (136), we have

$$\left(I - \frac{1\,1'}{N}\right)T = X\left(I - \frac{1\,1'}{m}\right), \tag{137}$$

or from (129)

$$\left(I - \frac{1\,1'}{N}\right)T = Y, \tag{138}$$

but as Y is the ipsative matrix obtained from X the matrix with equal means, Property 9.2 is proved.

Property 9.3

If the variances were not equated prior to ipsatizing, normalizing after ipsatizing will not make the resulting scores meaningful.

It was illustrated in Chapter II, and is considered to be self-evident, that ipsative scores are meaningful only if the means and variances of the underlying absolute or raw scores are equivalent. The development of Property 9.2 made it clear that failure to equate the means could be corrected by transforming the ipsative scores to deviation units. The question now is whether the effect of failing originally to equate the variances can be compensated for by an operation on the ipsative scores.

Property 9.3 implies the ipsatizing of a matrix such as G defined above, or at best G after transformation to deviation scores which we will call U as defined below:

$$U = \left(I - \frac{1\,1'}{N}\right)G. \tag{139}$$

Letting P be the ipsative matrix obtained from U we have

$$P = U\left(I - \frac{1\,1'}{m}\right). \tag{140}$$

The question is whether normalizing P will yield a matrix equivalent to Y. Letting D_{σ_P} be the diagonal matrix of standard deviations of P, then P is normalized by

$$PD_{\sigma_P}^{-1} = U\left(I - \frac{1\,1'}{m}\right)D_{\sigma_P}^{-1}, \tag{141}$$

or from (139) in (141)

$$PD_{\sigma_P}^{-1} = \left(I - \frac{1\,1'}{N}\right)G\left(I - \frac{1\,1'}{m}\right)D_{\sigma_P}^{-1}. \tag{142}$$

Now from (128) and (129) we observe that

$$Y = \left(I - \frac{1\,1'}{N}\right)GD_{\sigma}^{-1}\left(I - \frac{1\,1'}{m}\right). \tag{143}$$

From (142) and (143) it is clear that $PD_{\sigma_P}^{-1}$ will equal Y only if

$$\left(I - \frac{1\,1'}{m}\right)D_{\sigma_P}^{-1} = D_{\sigma}^{-1}\left(I - \frac{1\,1'}{m}\right),$$

but these two matrices are quite obviously not equal. Hence Property 9.3 is proved.

CHAPTER V

SUMMARY AND RECOMMENDATIONS

Summary

It was pointed out in the introduction that the properties of ipsative units are not well known. This study was designed to give a more complete and perhaps a better understanding of such units. In this chapter the properties of greatest significance will be reviewed and their consequences discussed. The numbers designating the properties coincide with that in Chapter IV.

Property 1.1

An ipsative intercorrelation matrix can be expressed as a simple function of the matrix of intercorrelations for the same variables prior to ipsatizing.

The implication of this property is that there is always a set of *raw* or *absolute* measures underlying the ipsative set. These measures may be very difficult or impossible to obtain, but nevertheless in theory they are there. Recognition of this fact should help the user of ipsative variables to avoid misinterpreting them.

Property 1.2

The sums of the columns (or rows) of an ipsative covariance matrix must always equal zero.

This property of ipsative variables can be put to work as an excellent intermediate check when computing intercorrelations for an ipsative set of data. It also, of course, implies that there will be a large number of negative values in any ipsative intercorrelation matrix.

Property 1.3

In the special case when the ipsative variances are equal, the sums of the columns (or rows) of the ipsative intercorrelation matrix are equal to zero.

This is a special case which in actual practice will probably never be encountered. Nevertheless, empirical observations in addition to those reported here have led the writer to believe that the column sums, of the intercorrelation matrices of the ipsative variables in common use, will approximate zero.

Property 1.4

Ipsative intercorrelation matrices are nonbasic or singular and thus have no regular inverse.

This is a very important property, and should be well understood by all researchers using ipsative variables. Much time and money can be expended to no avail in trying to find the *regular* inverse of an ipsative intercorrelation matrix. If an exact least-square solution for estimating some criterion is desired, the researcher must either delete a variable from the ipsative set or utilize the "general inverse" approach developed by Horst [9]. No modification of the ipsative intercorrelation matrix is necessary if iterative procedures are utilized. Iterative procedures may be utilized without altering the final multiple correlation whether or not a variable has been deleted from the original set. However, deleting a variable will usually reduce the labor involved in determining weights by iterative techniques.

Property 2.2

The sum of the covariance terms obtained between a specified criterion and a set of ipsative variables is zero.

Whatever the validity coefficients would have been for the underlying absolute scores of the traits corresponding to those being measured by a given ipsative set of variables, the resulting validity covariances will sum to zero. A knowledge of this property should assist the researcher to avoid false interpretations of the relation between a trait and some specified criterion. The researcher must always keep in mind the reservation voiced by Allport and Vernon [1] that these are *relative* not *absolute* measures.

Property 3.1

The multiple correlation of a set of ipsative variables with a criterion can be expressed as a function of the multiple correlation for the pre-ipsatized variables and their β coefficients, and except in one rare special case the ipsative multiple correlation is always less than the multiple correlation for the same variables prior to ipsatizing.

This property makes it clear that the researcher should strive to obtain *absolute* measures. Regardless of how adequately the job of prediction is performed by a set of ipsative variables, the predictions are as accurate as those possible with *absolute* measures of the same traits only in one rare, special case. Admittedly *absolute* measures may in some instance be more difficult to obtain than ipsative measures, but as long as the less difficult approach is not the optimal approach it can be considered only a temporary solution. Those interested in test construction should endeavor to develop techniques for obtaining absolute scales for measuring all attributes of behavior.

Property 4.1

The least-square estimate of a criterion using all of the variables of an ipsative set is identical with the least-square solution with any single variable deleted.

The discussion of Property 1.4 implied that a variable must be deleted from a complete ipsative set before a regular inverse exists for the intercorrelation matrix. Property 4.1 and its development in Chapter IV make it clear that the ability to predict any criterion is unaffected by the deletion of any single ipsative variable. This is true whether we delete the variable with the highest or lowest zero-order validity coefficient. Again this property should impress the test user and researcher with the highly dependent interrelationship of ipsative variables.

Property 6.1

When the variables in non-ipsative form are orthogonal, the ipsative intercorrelations will all be a negative constant value determined only by the number of variables.

This property serves to emphasize further the complex interdependency of ipsative variables even in this special case where the underlying absolute measures have zero correlations with each other. It also clearly illustrates the importance of the number of variables in determining the magnitude of the ipsative intercorrelations. In fact, in this special case the magnitude of the intercorrelations is determined exactly by the number of variables alone.

Property 6.2

When the variables in non-ipsative form are all correlated with each other to some constant degree a, the ipsative intercorrelations will be exactly the same as if the original variables had been orthogonal as in Property 6.1.

This property also illustrates the importance of the number of variables in determining ipsative intercorrelations. It also shows that the resulting ipsative intercorrelations are completely independent of the intercorrelations of the underlying absolute measures in the special case where these absolute measures are correlated with each other to some constant degree.

Property 6.3

When the column sums of the pre-ipsative intercorrelation matrix are all equal and the off-diagonal correlations are distributed normally, the proportion of negative values in the ipsative intercorrelation matrix can be determined readily from tabled values of the normal curve.

Although generally the properties of ipsative variables are not well known, the fact that ipsative intercorrelation matrices have a large number of negative values has long been recognized. The development relating to

Property 6.3 in Chapter IV makes it clear why there will always have to be a high percentage of negative correlations in an ipsative set. Furthermore, the development in Chapter IV and the illustration presented in Table 15 of Chapter III make it clear that the actual number of negative correlations can be determined quite accurately as a function of the number of variables and the variance of the intercorrelations. This property again demonstrates the complex interdependency of ipsative variables.

Property 7.1

When the original scores are in normalized units with a mean of zero and standard deviation of unity and the sums of the columns (or rows) of the original intercorrelation matrix are all equal, then the ipsative covariance matrix and the first-centroid residual are identical.

The discussion above has at several points implied that an ipsative set of variables contains less "information" than would the set of underlying absolute measures for the same traits. However, none of the above properties reveals the loss of "information" quite as clearly as Property 7.1. All researchers in psychology familiar with quantitative techniques are aware that the first centroid contains a tremendous portion of the relevant variance in any intercorrelation matrix. To state that the ipsative covariance matrix corresponds exactly (under the restrictions given) to the residual remaining after the removal of the first centroid from the intercorrelation matrix of the absolute measures, is the same as stating that a tremendous amount of "information" is missing in such an ipsative set. The example presented in Table 14 of Chapter III supports the hypothesis that this property will hold very well even when the restriction on the column sums of the original correlation matrix is not met.

It is not unreasonable to state, then, that ipsative covariance matrices contain essentially the same amount of "information" as the first-centroid residual obtainable from the intercorrelation matrix of the absolute measures for the same traits. Furthermore, the fact that this information is missing from an ipsative intercorrelation matrix will make it next to impossible to make anything psychologically meaningful out of a factor analysis of such data.

Property 8.1

Under a very special restriction a set of ipsative variables will correlate perfectly with their absolute counterparts.

This is another property bringing into focus the difference between ipsative variables and their underlying absolute counterparts, and it further illustrates the necessity of interpreting ipsative scores with caution. The statement of Allport and Vernon quoted in Chapter III emphasizes this need.

Property 9.1

If the column means of the pre-ipsative matrix are equal, the column means of the resulting ipsative matrix must be equal.

The discussion in Chapter II demonstrated that if the means and variances of a set of scores were not equated prior ot ipsatizing the resulting scores would have little meaning. Property 9.1 offers a fool-proof technique for checking on whether the means were equated prior to ipsatizing. This check is possible even when the ipsative scores are obtained directly and the underlying absolute scores are unknown.

This property further implies that no set of ipsative scores should be utilized in which the means are not equal or very nearly so. Fortunately, if only the means were different prior to ipsatizing and not the variances, then there is a transformation possible which will convert the scores to "true" ipsative units. This transformation is given in Property 9.2.

Property 9.2

If the variances but not the means were equated prior to ipsatizing, transforming the resulting ipsative scores to deviation units will yield interpretable results.

The development presented in support of Property 9.2 in Chapter IV makes it clear that if only the restriction on the means was not met prior to ipsatizing, then transforming the obtained scores into deviation units by variables will correct the fault. However, the further transformation of the obtained ipsative scores into standard units will serve no useful purpose.

Property 9.3

If the variances were not equated prior to ipsatizing, normalizing after ipsatizing will not make the resulting scores meaningful.

This property along with the discussion in Chapter II makes the burden placed on the shoulder of the ipsative test-maker clear. He must be as certain as he can possibly be that equal variances are maintained for the absolute scales underlying his ipsative scales even though he cannot observe these absolute measures. This seems to be somewhat of a paradox, but once recognized the clever test-maker will probably find ways of dealing with the problem. If he fails to do so the resulting ipsative measures will have little meaning even in the relative sense.

Recommendations

Not all of the properties discussed above lead immediately to recommendations. Some suggestions have already been made in the foregoing part of this chapter. Some of these suggestions can be stated more positively as recommendations. For instance, when computing intercorrelations for ipsative variables an extremely valuable check is to determine the sum of the variance of each variable in turn plus the covariances of that variable with each of the others. This sum should be exactly zero. In addition, when an external criterion is involved, the sum of its covariance terms with each of the ipsative variables should also be exactly zero within rounding error.

Some of the properties place restrictions on the use and analysis of ipsative data. Property 1.4 makes it clear, for example, that it is impossible to determine regression weights for a complete set of ipsative variables by calculating a *regular* inverse. However, generally the deletion of a variable from an ipsative set will make it possible to compute an inverse. It is recommended that a variable be deleted when it is desired to make predictions from an ipsative set.

Sometimes circumstances may arise where predictive weights are desired, but it does not seem wise to delete a variable arbitrarily. In these instances either of two approaches is recommended. The first is an iterative procedure developed by Horst [10]. In some instances this procedure may reach an optimal point prior to the selection of all the variables. For this reason and if an exact least-square solution is desired another method of determining regression weights is recommended. It also was developed by Horst [9], and is called the *general* inverse solution. The *general* inverse solution will yield weights for all of the variables in an ipsative set. This *general* inverse is not be to confused with the *regular* or usual inverse.

Property 7.1 implies that it will be extremely difficult, if not impossible, to obtain psychologically meaningful results from the factor analysis of a set of ipsative intercorrelations. It would seem that performing such an analysis would serve no purpose other than determining the rank of the matrix. For this reason, if such a set of data is factor analyzed, it is recommended that no attempt be made to rotate the resulting vectors to simple structure form.

All ipsative intercorrelation matrices are bound to have a large percentage of negative elements. This is made clear by Properties 6.1 and 6.3. Also from the first of these properties it is clear that for any symmetrical distribution the covariance terms will cluster around zero. This implies that in general ipsative intercorrleation matrices will have many near zero elements. It is strongly recommended that the values and even the signs of the correlations of *ipsative* attribute measures should not be confused with the values and signs which would have been obtained as correlations for *absolute* measures of the same attributes.

One of the first things to check when examining an ipsative instrument is the difference between each pair of means for the standardization population. Each of these differences should be zero. If they are not zero, it is recommended that the scores be transformed into deviation units before interpretation is made. If this operation is not performed errors in interpretation will result. This is true, as the discussion above particularly in Chapter II made clear, because ipsative scores can have meaning only if the underlying absolute measures have identical means and variances; furthermore, Property 9.1 states that if the means for the underlying absolute measures are equal then the means of the resulting ipsative measures *must* be equal. Property 9.2 shows that failure to equate the means can be corrected by transforming the ipsative scores into deviation units. On the other hand, Property 9.3 makes it clear that failure to equate the variances cannot be corrected by a transformation on the ipsative scores.

This statement leads naturally to another recommendation, i.e., that those constructing instruments which yield ipsative scores directly should exercise great caution in making as certain as possible that the variances for the underlying *absolute* variables are all equal. No suggestions as to how this can be done will be made in here but no doubt ideas which will contribute to satisfying this requirement will suggest themselves to test makers.

Property 8.1 should make it eminently clear to anyone using ipsative variables that the magnitude of such scores must never be confused with the magnitude of absolute measures for the same set of traits. Ipsative scores are relative scores. It is quite possible that a person obtaining a low ipsative score on a particular trait actually possesses more of the characteristic in question than a person obtaining a much higher ipsative score. It is imperative that users of ipsative variables interpret them in the relative sense only. It can further be recommended, because of Property 9.2 and the empirical examples of Chapter II, that when these interpretations are made it is important to use deviation scores, not normalized scores.

A number of the observations made in this study suggest that non-ipsative measures of a series of traits would be superior to ipsative scores because the former contain more information. The argument is mathematically sound, but the reader is cautioned that it is not ususally an easy task to develop "absolute" measures that correspond to the variables in an ipsative set. Indeed, it was the difficulty of obtaining valid "absolute" measures that led to the development of some of the available ipsative instruments. There is evidence that forced choice instruments yielding ipsative scores are not as influenced by the tendency to respond in the socially accepted manner and that they cannot be as easily faked as instruments designed to measure traits more directly.

The difficulty of obtaining directly measures that reflect precisely the absolute counterparts of ipsative measures is illustrated by the studies of Wright and Talbott. Wright [16] devised a set of rating scales designed to assess directly the traits included in the Edwards Personal Preference Schedule. Talbott [13] using Wright's scales compared their efficiency with the EPPS for predicting ten criteria. In eight instances the multiple correlation was higher using the EPPS than using the scales devised by Wright. This finding runs counter to what would be expected from the mathematical viewpoint developed here. But the mathematics is not threatened; instead this finding simply demonstrates the difficulty of developing "absolute" measures for some traits. Therefore, let there be no misunderstanding, some traits that can be relatively easily compared using ipsative techniques may be very difficult to assess validly using instruments designed to yield more direct or "absolute" measures. A set of measures said to yield scores that are the "absolute" counterparts of those yielded by an ipsative instrument should *not* be considered superior unless it can be demonstrated empirically that it does indeed contain more information.

REFERENCES

[1] Allport, G. W. and Vernon, P. E. *A study of values, manual of directions.* Boston: Houghton Mifflin, 1931.

[2] Allport, G. W., Vernon, P. E., and Lindzey, G. *A study of values, revised manual of directions.* Boston: Houghton Mifflin, 1951.

[3] Cattell, R. B. Psychological measurement: normative, ipsative, interactive. *Psychol. Rev.*, 1944, **51**, 292–303.

[4] Edwards, A. L. *Personal preference schedule: manual.* New York: Psychol. Corp., 1954.

[5] Ferguson, L. W., Humphreys, L. G., and Strong, F. W. A factorial analysis of interests and values. *J. educ. Psychol.*, 1941, **32**, 197–204.

[6] Guilford, J. P. When not to factor analyze. *Psychol. Bull.*, 1952, **49**, 31.

[7] Guttman, L. General theory and methods for matrix factoring. *Psychometrika,* 1944, **9**, 1–16.

[8] Harris, L. W. Relations among factors of raw, deviation, and double-centered score matrices. *J. exper. Educ.*, 1953, **22**, 53.

[9] Horst, P. *Matrix algebra for social scientists.* New York: Holt, Rinehart and Winston, 1963.

[10] Horst, P. The discrimination of two racial samples. *Psychometrika,* 1950, **15**, 271–289.

[11] Hotelling, H. Analysis of a complex of statistical variables into prinicpal components. *J. educ. Psychol.*, 1953, **24**, 417–441, 498–520.

[12] Kuder, F. G. *Kuder preference record, form C: Examiner's manual.* Chicago: Science Research Associates, 1946.

[13] Talbott, R. D. The multiple predictive efficiency of ipsative and normative personality measures. University of Washington. Contract No. Nonr-477 (08) and Public Health Grant No. M-743 (C3), February 1960.

[14] Thomson, G. H. *The factorial analysis of human ability.* New York: Houghton Mifflin, 1939.

[15] Thomson, G. H. *The factorial analysis of human ability.* New York: Houghton Mifflin, 1951.

[16] Wright, C. E. Relations between normative and ipsative measures of personality. University of Washington. Contract No. Nonr-477 (08) and Public Health Grant No. M-743 (C2), December 1957.

APPENDIX

TABLE 16

Raw Scores of the 129 Subjects on Six Predictor Variables and the Criterion

Subject	Predictor Variables* 1	2	3	4	5	6	Criterion† GPA
001	10	11	28	12	17	22	29
002	35	25	36	21	25	43	10
003	09	07	08	13	07	31	20
004	51	25	34	23	24	49	34
005	22	21	19	24	12	52	15
006	12	47	09	23	05	50	17
007	08	35	01	09	11	38	16
008	18	17	06	04	16	17	15
009	13	09	10	10	11	44	13
010	62	12	44	25	31	62	38
011	25	34	30	22	14	41	18
012	15	44	00	14	06	17	24
013	16	41	14	23	19	38	20
014	17	22	16	12	17	36	22
015	24	33	24	34	19	47	20
016	25	26	09	20	16	27	17
017	22	18	07	10	17	36	22
018	14	29	02	16	12	33	20
019	37	11	16	12	19	42	24
020	06	26	04	08	00	50	18
021	21	03	25	12	06	38	23
022	37	23	17	28	27	53	35
023	41	22	18	30	22	53	24
024	19	46	10	16	14	39	14
025	26	45	08	24	14	39	19
026	26	22	12	22	06	45	13
027	18	28	14	15	19	37	21
028	28	23	18	32	22	68	27
029	16	21	13	16	11	40	15
030	18	03	07	04	04	28	13
031	15	11	22	02	05	22	04
032	18	48	00	23	17	27	13
033	36	42	11	13	16	44	29
034	21	45	05	26	18	44	14
035	36	12	15	25	21	44	25
036	28	10	19	17	19	42	26
037	11	44	09	09	04	29	19
038	23	29	13	36	16	56	19
039	32	35	21	25	16	48	19
040	14	10	15	14	10	43	21
041	24	14	14	16	14	32	20
042	16	06	06	04	12	23	14
043	49	19	35	26	14	48	34
044	20	21	02	18	20	44	30
045	25	06	24	12	12	40	10
046	38	31	13	45	21	50	28
047	18	28	09	24	28	57	25
048	09	08	01	09	04	27	17
049	36	10	19	14	21	31	26
050	21	28	08	18	12	37	23
051	31	12	14	12	09	33	18
052	23	17	16	13	15	42	27
053	09	18	28	09	08	33	23
054	58	38	36	43	35	63	38
055	19	16	05	17	13	37	16
056	35	43	09	31	32	52	18
057	26	19	17	26	19	36	26
058	13	23	00	11	18	23	21
059	22	14	12	21	17	44	15
060	24	04	12	14	20	17	30
061	31	14	32	26	20	54	31
062	17	11	24	12	16	44	22
063	25	36	09	19	20	43	37
064	43	21	18	11	23	43	27
065	31	06	25	12	19	42	25
066	12	23	10	11	10	23	16
067	15	24	11	06	11	50	17
068	35	37	21	34	12	59	22
069	27	20	10	11	15	33	21
070	27	12	37	21	28	45	20
071	26	05	12	15	16	31	28
072	39	29	34	18	19	54	17
073	13	03	07	14	08	36	18
074	19	27	09	29	09	35	20
075	19	21	07	19	21	49	25
076	36	15	28	21	17	52	21
077	62	15	41	11	12	32	24
078	28	37	09	14	14	33	20

TABLE 16—(Continued)

Subject	Predictor Variables* 1	2	3	4	5	6	Criterion† GPA
079	26	37	07	24	18	45	29
080	15	38	16	16	16	55	23
081	18	10	14	10	18	50	19
082	13	40	05	28	09	44	18
083	09	21	04	07	06	26	12
084	28	39	09	24	23	44	14
085	34	06	09	07	11	27	31
086	29	11	07	15	12	47	13
087	22	19	13	30	20	45	26
088	28	44	24	24	08	41	19
089	07	17	21	11	09	41	21
090	30	01	12	10	14	48	27
091	33	41	22	22	31	51	21
092	12	26	21	15	15	49	23
093	36	34	27	30	35	51	35
094	31	23	21	22	14	47	32
095	35	18	16	23	18	53	25
096	12	24	10	09	04	34	18
097	31	25	20	29	18	36	24
098	19	37	03	24	27	52	26
099	48	50	24	41	28	61	36
100	29	12	32	16	15	43	20
101	20	30	12	11	15	45	19
102	22	16	14	03	17	37	17
103	15	07	37	16	07	44	18
104	08	28	11	28	16	56	18
105	39	23	17	30	31	46	22
106	15	33	33	35	11	56	13
107	27	00	29	15	14	40	23
108	19	06	37	06	17	48	19
109	43	36	40	14	27	38	22
110	20	33	02	15	17	47	14
111	24	05	14	16	09	32	20
112	32	15	11	06	16	38	27
113	10	02	18	05	04	33	17
114	06	13	06	06	04	29	21
115	11	14	10	16	10	46	14
116	01	25	08	27	08	26	17
117	10	13	12	10	14	37	27
118	13	41	10	35	16	50	24
119	34	39	27	37	26	48	29
120	10	21	00	05	06	27	17
121	18	40	05	24	10	44	10
122	19	39	02	21	07	35	10
123	12	22	06	18	16	54	15
124	29	18	16	23	25	57	25
125	10	22	18	23	08	44	25
126	19	27	05	16	13	39	24
127	31	21	09	15	23	47	26
128	17	26	00	11	06	30	08
129	35	17	39	14	21	52	30

* Predictor Variables:
1. Guilford-Zimmerman I, Vocabulary
2. Guilford-Zimmerman VII, Mechanical Knowledge
3. Cooperative English, Form OM, Spelling
4. Cooperative Mathematics, Form X, Part I
5. Cooperative Social Studies, Form X, Part II
6. A. C. E., Quantitative Section, 1948

† Decimals omitted

TABLE 17

Statistical Data for Empirical Example in Chapter III

Predictor Variables	r^{-1} 1	2	3	4	5	6	β_i	β_i^2/r_{ii}
1	1.881	.114	−.718	−.226	−.777	.093	.226	.027154
2	.114	1.520	.423	−.832	−.104	−.018	−.151	.015001
3	−.718	.423	1.637	−.049	−.037	−.401	.021	.000269
4	−.226	−.832	−.049	2.170	−.359	−.815	.118	.006417
5	−.777	−.104	−.037	−.359	1.847	−.385	.359	.069779
6	.093	−.018	−.401	−.815	−.385	1.792	.012	.000080